Area minimizing surfaces for singular boundary values

Paul Creutz

(joint work with Martin Fitzi)

University of Cologne

October 1, 2020

1. Classical Plateau-Douglas problem

Standing assumptions

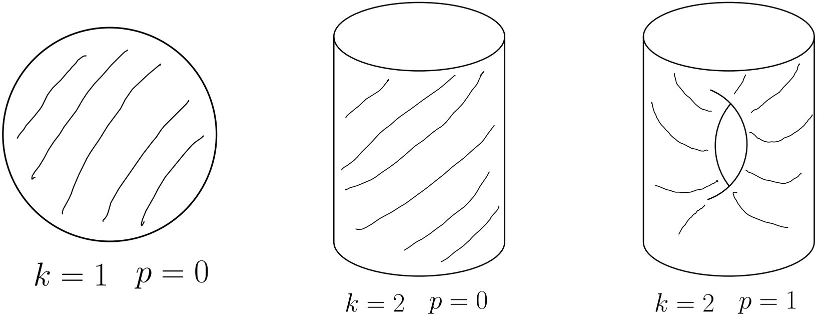
- Fix a 2-dimensional differentiable manifold S which is...
  - compact,
  - connected,
  - orientable,
  - and has boundary.

Standing assumptions

- Fix a 2-dimensional differentiable manifold S which is...
  - compact,
  - connected,
  - orientable,
  - and has boundary.
- S uniquely determined by...
  - $k :=$ number of boundary components, and
  - $p :=$ genus of S.

Standing assumptions

- S compact, connected, orientable surface.
- $k = \sharp$ boundary components, $\quad p =$ genus.



Standing assumptions

- S compact, connected, orientable surface with k boundary components.
- $\Gamma \subset \mathbb{R}^n$ a configuration of k disjoint Jordan curves.

Motivation

- S compact, connected, orientable surface with k boundary components.
- $\Gamma \subset \mathbb{R}^n$ a configuration of k disjoint Jordan curves .

Motivating Question

Is there a minimal surface $T_{\min} \subset \mathbb{R}^n$ s.t.

- $\partial T_{\min} = \Gamma$ and
- $T_{\min} \simeq S$?

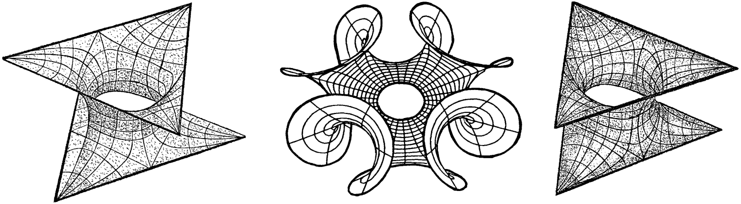


Figure taken from Dierkes-Hildebrandt-Sauvigny "Minimal surfaces"

The area functional

- Sobolev space $W^{1,2}(S, \mathbb{R}^n)$ defined by

$$u \in W^{1,2}(S, \mathbb{R}^n) :\Leftrightarrow u \in L^2 \wedge Du \in L^2.$$

The area functional

- Sobolev space $W^{1,2}(S, \mathbb{R}^n)$ defined by

$$u \in W^{1,2}(S, \mathbb{R}^n) :\Leftrightarrow u \in L^2 \wedge Du \in L^2.$$

- For $u \in W^{1,2}(S, \mathbb{R}^n)$ define

$$\mathrm{Area}(u) := \int_S \sqrt{\det\left((D_p u)^* \cdot (D_p u)\right)}\,\mathrm{d}p$$

The area functional

- Sobolev space $W^{1,2}(S,\mathbb{R}^n)$ defined by

$$u \in W^{1,2}(S,\mathbb{R}^n) :\Leftrightarrow u \in L^2 \wedge Du \in L^2.$$

- For $u \in W^{1,2}(S,\mathbb{R}^n)$ define

$$\mathrm{Area}(u) := \int_S \sqrt{\det\left((D_p u)^* \cdot (D_p u)\right)}\,\mathrm{d}p$$

- By the area formula

$$\mathrm{Area}(u) = \int_{\mathbb{R}^n} \mathrm{card}\{u^{-1}(y)\}\,\mathrm{d}\mathcal{H}^2(y).$$

Plateau-Douglas problem

Motivating Question: Is there a minimal surface $T_{\min} \subset \mathbb{R}^n$ s.t.

- $\partial T_{\min} = \Gamma$ and
- $T_{\min} \simeq S$?

- Define set of *fillings* as

$$\Lambda(S, \Gamma, \mathbb{R}^n) := \{u \in W^{1,2}(S, \mathbb{R}^n) : u|_{\partial S} \text{ parametrizes } \Gamma\}.$$

Plateau-Douglas problem

Motivating Question: Is there a minimal surface $T_{\min} \subset \mathbb{R}^n$ s.t.

- $\partial T_{\min} = \Gamma$ and
- $T_{\min} \simeq S$?

- Define set of *fillings* as

$$\Lambda(S, \Gamma, \mathbb{R}^n) := \{u \in W^{1,2}(S, \mathbb{R}^n) : u|_{\partial S} \text{ parametrizes } \Gamma\}.$$

- $u_{\min} \in \Lambda(M, \Gamma, \mathbb{R}^n)$ is called *area minimizer* if

$$\text{Area}(u_{\min}) \leq \text{Area}(u) \qquad ; \forall u \in \Lambda(S, \Gamma, \mathbb{R}^n).$$

Plateau-Douglas problem

Motivating Question: Is there a minimal surface $T_{\min} \subset \mathbb{R}^n$ s.t.

- $\partial T_{\min} = \Gamma$ and
- $T_{\min} \simeq S$?

- Define set of *fillings* as

$$\Lambda(S, \Gamma, \mathbb{R}^n) := \{u \in W^{1,2}(S, \mathbb{R}^n) : u|_{\partial S} \text{ parametrizes } \Gamma\}.$$

- $u_{\min} \in \Lambda(M, \Gamma, \mathbb{R}^n)$ is called *area minimizer* if

$$\text{Area}(u_{\min}) \leq \text{Area}(u) \qquad ; \forall u \in \Lambda(S, \Gamma, \mathbb{R}^n).$$

Question (Plateau-Douglas problem)

- *Is there an area minimizer $u_{\min} \in \Lambda(S, \Gamma, \mathbb{R}^n)$?*
- *If yes: Regularity of $u_{\min}$?*

Existence & Regularity of area minimizers

Set $\Lambda(S, \Gamma, \mathbb{R}^n) := \{u \in W^{1,2}(S, \mathbb{R}^n) : u|_{\partial S} \text{ parametrizes } \Gamma\}$.

- Is there an area minimizer $u_{\text{min}} \in \Lambda(S, \Gamma, \mathbb{R}^n)$?
- If yes: Regularity of u_{min}?

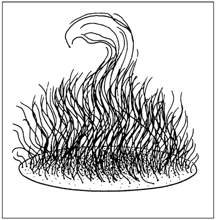
- Existence: **Yes!** (nontrivial)

Existence & Regularity of area minimizers

Set $\Lambda(S,\Gamma,\mathbb{R}^n) := \{u \in W^{1,2}(S,\mathbb{R}^n) : u|_{\partial S} \text{ parametrizes } \Gamma\}$.

- Is there an area minimizer $u_{\min} \in \Lambda(S,\Gamma,\mathbb{R}^n)$?
- If yes: Regularity of $u_{\min}$?

- Existence: **Yes!** (nontrivial)
- However: <u>In general no reasonable regularity!</u>



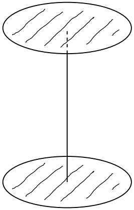


Figure taken from Dierkes-Hildebrandt-Sauvigny
"Minimal surfaces"

Energy functional

- Set
$$\mathcal{R}(S) := \{\text{Riemannian metrics on } S\}.$$
- For $g \in \mathcal{R}(S)$ and $u \in W^{1,2}(S, \mathbb{R}^n)$ define *energy*
$$E_+^2(u, g) := \int_S \max_{v \in T_p S} \frac{|D_p u(v)|^2}{|v|_g^2} \, \mathrm{d}A_g(p).$$

Energy vs. Area

- $E_+^2(u,g) = \int_S \max_{v \in T_pS} \frac{|D_pu(v)|^2}{|v|_g^2} \, \mathrm{d}A_g(p)$

Let $u \in W^{1,2}(S, \mathbb{R}^n)$.

- $\mathrm{Area}(u) \leq E_+^2(u,g)$ for all $g \in \mathcal{R}(S)$.

Energy vs. Area

- $E^2_+(u,g) = \int_S \max_{v\in T_pS} \frac{|D_pu(v)|^2}{|v|^2_g} \,\mathrm{d}A_g(p)$

Let $u \in W^{1,2}(S,\mathbb{R}^n)$.

- $\mathrm{Area}(u) \leq E^2_+(u,g)$ for all $g \in \mathcal{R}(S)$.
- $\mathrm{Area}(u) = \inf_{g\in\mathcal{R}(S)} E^2_+(u,g)$, (Fitzi-Wenger, 2019).

Energy vs. Area

- $E_+^2(u,g) = \int_S \max_{v \in T_pS} \frac{|D_pu(v)|^2}{|v|_g^2} \, \mathrm{d}A_g(p)$

Let $u \in W^{1,2}(S, \mathbb{R}^n)$.

- $\text{Area}(u) \leq E_+^2(u,g)$ for all $g \in \mathcal{R}(S)$.
- $\text{Area}(u) = \inf_{g \in \mathcal{R}(S)} E_+^2(u,g)$, (Fitzi-Wenger, 2019).
- The following are equivalent:
  - $\text{Area}(u) = E_+^2(u,g)$.
  - $u : (S,g) \to \mathbb{R}^n$ is weakly conformal.
  - $E_+^2(u,g) \leq E_+^2(u,h)$ for all $h \in \mathcal{R}(S)$.

Energy minimizers

- $\text{Area}(u) = \inf_{g \in \mathcal{R}(S)} E_+^2(u, g)$

- $(u_{\min}, g_{\min})$ where $u_{\min} \in \Lambda(S, \Gamma, \mathbb{R}^n)$ and $g_{\min} \in \mathcal{R}(S)$ is called *energy minimizing* pair if

$$E_+^2(u_{\min}, g_{\min}) \leq E_+^2(u, g) \quad ; \forall u \in \Lambda(S, \Gamma, \mathbb{R}^n); \forall g \in \mathcal{R}(S).$$

Energy minimizers

- $\text{Area}(u) = \inf_{g \in \mathcal{R}(S)} E_+^2(u, g)$

- $(u_{\min}, g_{\min})$ where $u_{\min} \in \Lambda(S, \Gamma, \mathbb{R}^n)$ and $g_{\min} \in \mathcal{R}(S)$ is called *energy minimizing* pair if

$$E_+^2(u_{\min}, g_{\min}) \leq E_+^2(u, g) \quad ; \forall u \in \Lambda(S, \Gamma, \mathbb{R}^n); \forall g \in \mathcal{R}(S).$$

$(u_{\min}, g_{\min})$ energy minimizing $\rightsquigarrow$ **$u_{\min}$ area minimizer**

Energy minimizers

- $\text{Area}(u) = \inf_{g \in \mathcal{R}(S)} E_+^2(u, g)$

- $(u_{\min}, g_{\min})$ where $u_{\min} \in \Lambda(S, \Gamma, \mathbb{R}^n)$ and $g_{\min} \in \mathcal{R}(S)$ is called *energy minimizing* pair if

$$E_+^2(u_{\min}, g_{\min}) \leq E_+^2(u, g) \quad ; \forall u \in \Lambda(S, \Gamma, \mathbb{R}^n); \forall g \in \mathcal{R}(S).$$

$(u_{\min}, g_{\min})$ energy minimizing $\rightsquigarrow$ **$u_{\min}$ area minimizer**

Question

- *Are there $(u_{\min}, g_{\min})$ energy minimizing?*
- *If yes: Regularity of $u_{\min}$?*

Regularity of energy minimizers

- Are there $(u_{\min}, g_{\min})$ energy minimizing?
- If yes: Regularity of $u_{\min}$?

If $(u_{\min}, g_{\min})$ is energy minimizing

$\rightsquigarrow$ $u_{\min} : (S, g_{\min}) \to \mathbb{R}^n$ is...

- weakly conformal,
- harmonic,
- smooth, and
- a branched immersion.

Existence of energy minimizers

Are there $(u_{\min}, g_{\min})$ energy minimizing?

Existence of energy minimizers

Are there $(u_{\text{min}}, g_{\text{min}})$ energy minimizing?

In general: No!

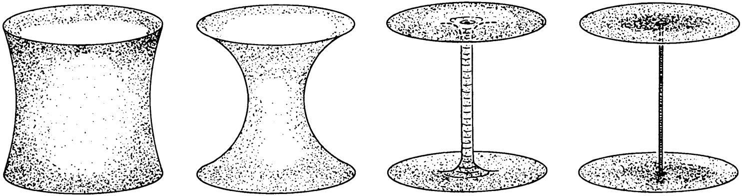


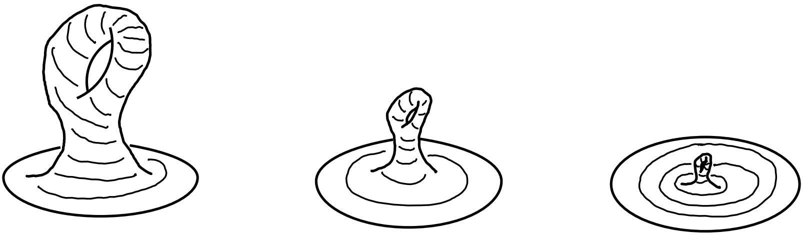
Figure taken from Dierkes-Hildebrandt-Tromba "Global analysis of minimal surfaces"

Existence of energy minimizers

Are there $u_{\min} \in \Lambda(M, \Gamma, \mathbb{R}^n)$ and $g_{\min} \in \mathcal{R}(S)$ s.t.

$$E_+^2(u_{\min}, g_{\min}) \leq E_+^2(u, g) \qquad ; \forall u \in \Lambda(S, \Gamma, \mathbb{R}^n); \forall g \in \mathcal{R}(S)?$$

In general: No!



Reductions

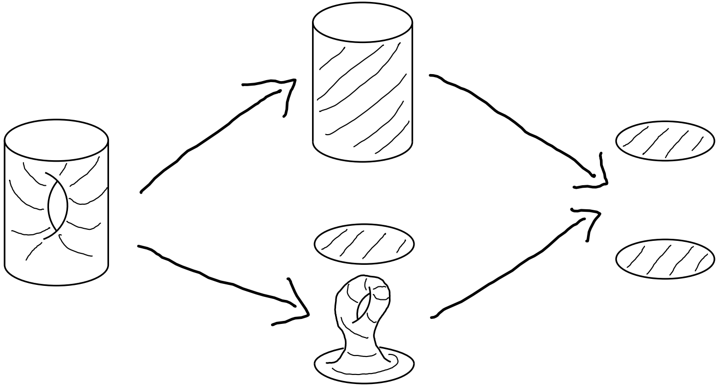
S^* orientable compact surface (possibly disconnected!).

- S^* is called a *reduction* of S if
  - S^* has k boundary components,
  - $\text{genus}(S^*) \leq \text{genus}(S)$, and
  - either S^* is disconnected or $\text{genus}(S^*) < \text{genus}(S)$.

Reductions

S^* is called a *reduction of S* if

- S^* has k boundary components,
- $\text{genus}(S^*) \leq \text{genus}(S)$, and
- either S^* is disconnected or $\text{genus}(S^*) < \text{genus}(S)$.



Douglas condition

S^* is called a *reduction of* S if

1. S^* has k boundary components,
2. $\text{genus}(S^*) \leq \text{genus}(S)$, and
3. either S^* is disconnected or $\text{genus}(S^*) < \text{genus}(S)$.

The *Douglas condition* holds if

$$\inf_{u \in \Lambda(S,\Gamma,\mathbb{R}^n)} \text{Area}(u) < \inf_{u^* \in \Lambda(S^*,\Gamma,\mathbb{R}^n)} \text{Area}(u^*)$$

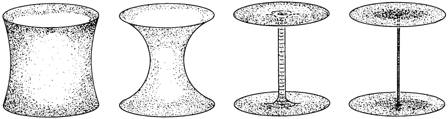
for every reduction S^* of S.

Douglas condition

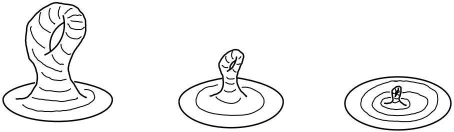
The *Douglas condition* holds if

$$\inf_{u \in \Lambda(S,\Gamma,\mathbb{R}^n)} \mathrm{Area}(u) < \inf_{u^* \in \Lambda(S^*,\Gamma,\mathbb{R}^n)} \mathrm{Area}(u^*)$$

for every reduction S^* of S.



$\rightsquigarrow$ Douglas condition violated for $S^* = \mathbb{D} \sqcup \mathbb{D}$.



$\rightsquigarrow$ Douglas condition violated for $S^* = \mathbb{D}$.

Douglas result

The *Douglas condition* holds if

$$\inf_{u \in \Lambda(S,\Gamma,\mathbb{R}^n)} \text{Area}(u) < \inf_{u^* \in \Lambda(S^*,\Gamma,\mathbb{R}^n)} \text{Area}(u^*)$$

for every reduction S^* of S.

Theorem (Douglas '39)

If Douglas condition holds $\rightsquigarrow \exists (u_{\min}, g_{\min})$ *energy minimizing.*

Douglas result

Theorem (Douglas '39)

If Douglas condition holds $\rightsquigarrow \exists (u_{\min}, g_{\min})$ *energy minimizing.*

Corollary

If Douglas condition holds $\rightsquigarrow \exists$ *area minimizer* $u_{\min}$ *s.t.*

- $u_{\min}$ *is a branched immersion, and*
- $\exists g \in \mathcal{R}(S)$ *s.t.* $u_{\min} : (S, g) \to \mathbb{R}^n$ *is weakly conformal and harmonic.*

2. Plateau-Douglas problem for singular boundary values

Standing assumptions

Classical problem:

- S compact, connected, orientable surface with k boundary components.
- $\Gamma \subset \mathbb{R}^n$ a configuration of k disjoint Jordan curves.

Standing assumptions

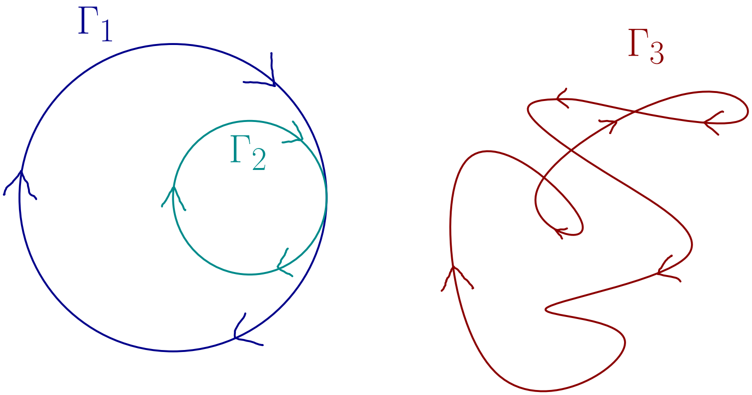
Now:

- S compact, connected, orientable surface with k boundary components.
- $\Gamma \subset \mathbb{R}^n$ a configuration of k ~~disjoint Jordan~~ curves.

Standing assumptions

- S compact, connected, orientable surface with k boundary components.
- $\Gamma \subset \mathbb{R}^n$ a configuration of k closed curves.

$$\Gamma = (\Gamma_1, \Gamma_2, \Gamma_3)$$



Plateau-Douglas problem for singular boundary values

- S compact, connected, orientable surface with k boundary components.
- $\Gamma \subset \mathbb{R}^n$ a configuration of k closed curves.

- Define set of *fillings* as

$$\Lambda(S, \Gamma, \mathbb{R}^n) := \{u \in W^{1,2}(S, \mathbb{R}^n) : u|_{\partial S} \text{ parametrizes } \Gamma\}.$$

Plateau-Douglas problem for singular boundary values

- S compact, connected, orientable surface with k boundary components.
- $\Gamma \subset \mathbb{R}^n$ a configuration of k closed curves.

- Define set of *fillings* as

$$\Lambda(S, \Gamma, \mathbb{R}^n) := \{u \in W^{1,2}(S, \mathbb{R}^n) : u|_{\partial S} \text{ parametrizes } \Gamma\}.$$

Question (PD problem for singular boundary values)

If Douglas-condition holds

$\rightsquigarrow \exists$ *'reasonably regular' area minimizer* $u_{\min} \in \Lambda(S, \Gamma, \mathbb{R}^n)$*?*

Existence of energy minimizers?

- S compact, connected, orientable surface with k boundary components.
- $\Gamma \subset \mathbb{R}^n$ a configuration of k closed curves.

- $(u_{\min}, g_{\min})$ is called *energy minimizing* pair if

$$E^2_+(u_{\min}, g_{\min}) \leq E^2_+(u, g) \quad ; \forall u \in \Lambda(S, \Gamma, \mathbb{R}^n); \forall g \in \mathcal{R}(S).$$

Existence of energy minimizers?

- S compact, connected, orientable surface with k boundary components.
- $\Gamma \subset \mathbb{R}^n$ a configuration of k closed curves.

- $(u_{\min}, g_{\min})$ is called *energy minimizing* pair if

$$E^2_+(u_{\min}, g_{\min}) \leq E^2_+(u, g) \quad ; \forall u \in \Lambda(S, \Gamma, \mathbb{R}^n); \forall g \in \mathcal{R}(S).$$

Question

If Douglas-condition holds

$\rightsquigarrow \exists (u_{\min}, g_{\min})$ *energy minimizing?*

Existence of energy minimizers?

Question

If Douglas condition holds $\rightsquigarrow \exists(u_{\mathrm{min}}, g_{\mathrm{min}})$ *energy minimizing?*

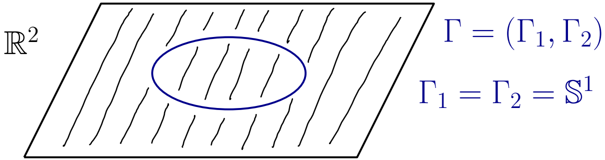
In general no!

Existence of energy minimizers?

Question

If Douglas condition holds $\rightsquigarrow \exists (u_{\mathrm{min}}, g_{\mathrm{min}})$ *energy minimizing?*

In general no!

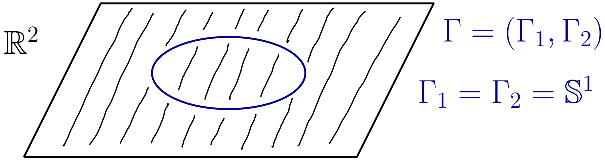


Existence of energy minimizers?

Question

If Douglas condition holds $\rightsquigarrow \exists(u_{\min}, g_{\min})$ *energy minimizing?*

In general no!



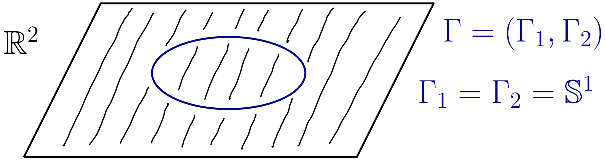
$$S = \mathbb{S}^1 \times [0,1] \rightsquigarrow \inf_{u \in \Lambda(S,\Gamma,X)} \mathrm{Area}(u) = 0$$

Existence of energy minimizers?

Question

If Douglas condition holds $\rightsquigarrow \exists(u_{\min}, g_{\min})$ *energy minimizing?*

In general no!



$$S = \mathbb{S}^1 \times [0,1] \rightsquigarrow \inf_{u \in \Lambda(S,\Gamma,X)} \mathrm{Area}(u) = 0$$

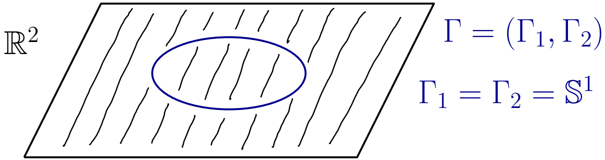
$$S^* \text{ reduction} \rightsquigarrow S^* = \mathbb{D} \sqcup \mathbb{D} \rightsquigarrow \inf_{u \in \Lambda(S^*,\Gamma,X)} \mathrm{Area}(u^*) = 2 \cdot \pi$$

Existence of energy minimizers?

Question

If Douglas condition holds $\rightsquigarrow \exists(u_{\min}, g_{\min})$ *energy minimizing?*

In general no!



$$S = \mathbb{S}^1 \times [0,1] \rightsquigarrow \inf_{u \in \Lambda(S,\Gamma,X)} \mathrm{Area}(u) = 0$$

$$S^* \text{ reduction} \rightsquigarrow S^* = \mathbb{D} \sqcup \mathbb{D} \rightsquigarrow \inf_{u \in \Lambda(S^*,\Gamma,X)} \mathrm{Area}(u^*) = 2 \cdot \pi$$

If $(u_{\min}, g_{\min})$ energy minimizing for (S, Γ)

$\rightsquigarrow$ $u_{\min}$ weakly conformal + $\mathrm{Area}(u_{\min}) = 0$

$\rightsquigarrow$ $u_{\min}$ constant $\lightning$

Previous results

Question (PD problem for singular boundary values)

If Douglas-condition holds
$\rightsquigarrow \exists$ *'reasonably regular' area minimizer* $u_{\min} \in \Lambda(S, \Gamma, \mathbb{R}^n)$*?*

- "Yes" for $S = \mathbb{D}$ (Hass '91):
  - $u_{\min} \in C^0(S, \mathbb{R}^n)$.
  - $u_{\min}$ a branched immersion on $S \setminus u^{-1}(\Gamma)$.
  - Caution: In general $u \notin \Lambda(S, \Gamma, \mathbb{R}^n)$.

Previous results

Question (PD problem for singular boundary values)

If Douglas-condition holds
$\rightsquigarrow \exists$ *'reasonably regular' area minimizer* $u_{\min} \in \Lambda(S, \Gamma, \mathbb{R}^n)$*?*

- "Yes" for $S = \mathbb{D}$ (Hass '91):
  - $u_{\min} \in C^0(S, \mathbb{R}^n)$.
  - $u_{\min}$ a branched immersion on $S \setminus u^{-1}(\Gamma)$.
  - Caution: In general $u \notin \Lambda(S, \Gamma, \mathbb{R}^n)$.
- Additionally assuming "Condition of adhesion"

  $\rightsquigarrow$ existence of energy minimizing $(u_{\min}, g_{\min})$ (Iseri '96).

Main result

Question (PD problem for singular boundary values)

If Douglas-condition holds
$\rightsquigarrow \exists$ *'reasonably regular' area minimizer* $u_{\min} \in \Lambda(S, \Gamma, \mathbb{R}^n)$*?*

Theorem (C.-Fitzi, 2020)

Yes!

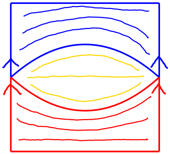
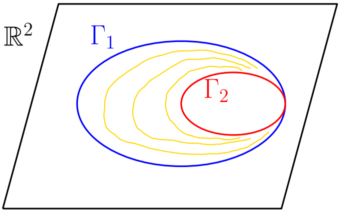
- *There is* $g \in \mathcal{R}(S)$ *such that* $u_{\min} : (S, g) \to \mathbb{R}^n$ *is a* ***weakly conformal, harmonic,*** *branched immersion on* $S \setminus u^{-1}(\Gamma)$.
- *If* Γ *is* $C^2 \rightsquigarrow u_{\min}$ *locally Lipschitz on* $S \setminus \partial S$.
- $u_{\min}$ *is Hoelder continuous (up to the boundary).*

Main result

If Douglas-condition holds $\rightsquigarrow \exists$ area minimizer $u_{\text{min}} \in \Lambda(S, \Gamma, X)$ s.t.

- $\exists g \in \mathcal{R}(S)$ s.t. $u_{\text{min}} : (S, g) \to \mathbb{R}^n$ is a weakly conformal, harmonic, branched immersion on $S \setminus u^{-1}(\Gamma)$.
- If Γ is $C^2 \rightsquigarrow u_{\text{min}}$ locally Lipschitz on $S \setminus \partial S$.
- u_{min} is Hoelder continuous (up to the boundary).

$\Gamma = (\Gamma_1, \Gamma_2)$

$S = \mathbb{S}^1 \times [0,1] =$  $\mapsto$ 

3. Plateau-Douglas problem in singular spaces

Standing assumptions

Classical problem:

- S compact, connected, orientable surface with k boundary components.
- $\Gamma \subset \mathbb{R}^n$ a configuration of k disjoint Jordan curves.

Standing assumptions

Now:

- S compact, connected, orientable surface with k boundary components.
- $X = (X, d)$ a proper metric space.
- $\Gamma \subset X$ a configuration of k disjoint Jordan curves.

Geometric analyis in metric spaces

- S compact, connected, orientable surface with k boundary components.
- $X = (X, d)$ a proper metric space.
- $\Gamma \subset X$ a configuration of k disjoint Jordan curves.

- $u \in W^{1,2}(S, X)$ if
  - for all $x \in X$ one has $u_x := d(x, u(-)) \in W^{1,2}(S, \mathbb{R})$, and
  - for fixed $g \in \mathcal{R}(S)$ the energy $E_+^2(u_x, g)$ is uniformly bounded.

Geometric analyis in metric spaces

- S compact, connected, orientable surface with k boundary components.
- $X = (X, d)$ a proper metric space.
- $\Gamma \subset X$ a configuration of k disjoint Jordan curves.

- $u \in W^{1,2}(S, X)$ if
  - for all $x \in X$ one has $u_x := d(x, u(-)) \in W^{1,2}(S, \mathbb{R})$, and
  - for fixed $g \in \mathcal{R}(S)$ the energy $E_+^2(u_x, g)$ is uniformly bounded.
- Trace $u|_{\partial S}$ defined for $u \in W^{1,2}(S, X)$.

$\rightsquigarrow$ $\Lambda(S, \Gamma, X) := \{u \in W^{1,2}(S, X) : \ u|_{\partial S} \text{ parametrizes } \Gamma\}$

Geometric analysis in metric spaces

- For $u \in W^{1,2}(S, X)$ define

$$\text{Area}(u) := \int_X \text{card}\{u^{-1}(y)\} \, \mathrm{d}\mathcal{H}^2(y).$$

- Also $E_+^2(u, g)$ defined for $u \in W^{1,2}(S, X)$ and $g \in \mathcal{R}(S)$.

Plateau-Douglas problem in metric spaces

- S compact, connected, orientable surface with k boundary components.
- $X = (X, d)$ a proper metric space.
- $\Gamma \subset X$ a configuration of k disjoint Jordan curves.

- $u_{\min} \in \Lambda(M, \Gamma, X)$ is called *area minimizer* if

$$\operatorname{Area}(u_{\min}) \leq \operatorname{Area}(u) \qquad ; \forall u \in \Lambda(S, \Gamma, X).$$

Question (PD problem for singular spaces)

If Douglas-condition holds

$\rightsquigarrow \exists$ *'reasonably regular' area minimizer* $u_{\min} \in \Lambda(S, \Gamma, \mathbb{R}^n)$*?*

Energy minimizers are not(!) area minimizers

- ~~$\mathrm{Area}(u) = \inf_{g \in \mathcal{R}(S)} E_+^2(u,g)$~~

- $(u_{\min}, g_{\min})$ where $u_{\min} \in \Lambda(S, \Gamma, X)$ and $g_{\min} \in \mathcal{R}(S)$ is called *energy minimizing* pair if

$$E_+^2(u_{\min}, g_{\min}) \leq E_+^2(u,g) \quad ; \forall u \in \Lambda(S, \Gamma, X); \forall g \in \mathcal{R}(S).$$

$(u_{\min}, g_{\min})$ energy minimizing $\quad \not\rightsquigarrow \quad$ **$u_{\min}$ area minimizer**

Replacement for conformality

For $u \in W^{1,2}(S, \mathbb{R}^n)$ the following are equivalent:

- $\text{Area}(u) = E_+^2(u, g)$.
- $u : (S, g) \to \mathbb{R}^n$ is weakly conformal.
- $E_+^2(u, g) \leq E_+^2(u, h)$ for all $h \in \mathcal{R}(S)$.

- We say that $u : (S, g) \to X$ is *infinitesimally isotropic* if

$$E_+^2(u, g) \leq E_+^2(u, h) \qquad ; \forall h \in \mathcal{R}(S).$$

Plateau-Douglas problem in metric spaces

- $X = (X, d)$ a proper metric space.
- $\Gamma \subset X$ a configuration of k disjoint Jordan curves.
- $u : (S, g) \to X$ is *infinitesimally isotropic* if $E_+^2(u, g) \leq E_+^2(u, h); \forall h \in \mathcal{R}(S)$.

Theorem (C.-Fitzi, 2020)

If Douglas-condition holds

$\rightsquigarrow \exists$ *area minimizer* $u_{\min} \in \Lambda(S, \Gamma, X)$ *and* $\exists g \in \mathcal{R}(S)$ *s.t.*

$u_{\min} : (S, g) \to X$ *is infinitesimally isotropic.*

Plateau-Douglas problem in metric spaces

- $X = (X, d)$ a proper metric space.
- $\Gamma \subset X$ a configuration of k disjoint Jordan curves.
- $u : (S, g) \to X$ is *infinitesimally isotropic* if $E_+^2(u, g) \leq E_+^2(u, h); \forall h \in \mathcal{R}(S)$.

Theorem (C.-Fitzi, 2020)

If Douglas-condition holds

$\rightsquigarrow \exists$ *area minimizer* $u_{\min} \in \Lambda(S, \Gamma, X)$ *and* $\exists g \in \mathcal{R}(S)$ *s.t.*

$u_{\min} : (S, g) \to X$ *is infinitesimally isotropic.*

- Generalizes previous results of Jost('85) and Fitzi-Wenger(2019).
- New e.g. for general complete Riemannian manifolds X.

4. Proof sketch

Statements

- $\Gamma \subset \mathbb{R}^n$ a configuration of k closed curves.

Theorem 1

If Douglas-condition holds

$\rightsquigarrow \exists$ 'reasonable' area minimizer $u_{\text{min}} \in \Lambda(S, \Gamma, \mathbb{R}^n)$

Statements

- $\Gamma \subset \mathbb{R}^n$ a configuration of k closed curves.

Theorem 1

If Douglas-condition holds

$\rightsquigarrow \exists$ 'reasonable' area minimizer $u_{\min} \in \Lambda(S, \Gamma, \mathbb{R}^n)$

- X a proper metric space,
- $\Gamma \subset X$ a configuration of k disjoint Jordan curves.

Theorem 2

If Douglas-condition holds

$\rightsquigarrow \exists$ 'reasonable' area minimizer $u_{\min} \in \Lambda(S, \Gamma, X)$

Proof of Theorem 1 (modulo Theorem 2)

Idea:

- Given $X = \mathbb{R}^n$ regular and $\Gamma \subset X$ singular

Proof of Theorem 1 (modulo Theorem 2)

Idea:

- Given $X = \mathbb{R}^n$ regular and $\Gamma \subset X$ singular
- Construct $\widetilde{X}$ singular and $\widetilde{\Gamma} \subset \widetilde{X}$ regular s.t.

$$\text{PD problem for } (X, \Gamma) \leftrightsquigarrow \text{PD problem for } (\widetilde{X}, \widetilde{\Gamma}).$$

Proof of Theorem 1 (modulo Theorem 2)

Idea:

- Given $X = \mathbb{R}^n$ regular and $\Gamma \subset X$ singular
- Construct $\widetilde{X}$ singular and $\widetilde{\Gamma} \subset \widetilde{X}$ regular s.t.

$$\text{PD problem for } (X, \Gamma) \leftrightsquigarrow \text{PD problem for } (\widetilde{X}, \widetilde{\Gamma}).$$

- Apply Theorem 2 to solve PD problem for $(\widetilde{X}, \widetilde{\Gamma})$. $\square$

The construction

Given $X = \mathbb{R}^n$ regular and $\Gamma \subset X$ singular .

$\rightsquigarrow$ Construct $\widetilde{X}$ singular and $\widetilde{\Gamma} \subset \widetilde{X}$ regular s.t.

PD problem for $(X, \Gamma) \leftrightsquigarrow$ PD problem for $(\widetilde{X}, \widetilde{\Gamma})$.

The construction

Given $X = \mathbb{R}^n$ regular and $\Gamma \subset X$ singular .
$\rightsquigarrow$ Construct $\widetilde{X}$ singular and $\widetilde{\Gamma} \subset \widetilde{X}$ regular s.t.
PD problem for $(X, \Gamma) \leftrightsquigarrow$ PD problem for $(\widetilde{X}, \widetilde{\Gamma})$.

- W.l.o.g. $\Gamma = (\Gamma_1, ..., \Gamma_k)$ with $\ell(\Gamma_i) \leq 2\pi$.

The construction

Given $X = \mathbb{R}^n$ regular and $\Gamma \subset X$ singular .
$\rightsquigarrow$ Construct $\widetilde{X}$ singular and $\widetilde{\Gamma} \subset \widetilde{X}$ regular s.t.
PD problem for $(X, \Gamma) \leftrightsquigarrow$ PD problem for $(\widetilde{X}, \widetilde{\Gamma})$.

- W.l.o.g. $\Gamma = (\Gamma_1, ..., \Gamma_k)$ with $\ell(\Gamma_i) \leq 2\pi$.
- $\gamma_i : \mathbb{S}^1 \to \Gamma_i$ constant speed parametrization.

The construction

Given $X = \mathbb{R}^n$ regular and $\Gamma \subset X$ singular .

$\rightsquigarrow$ Construct $\widetilde{X}$ singular and $\widetilde{\Gamma} \subset \widetilde{X}$ regular s.t.

PD problem for $(X, \Gamma) \leftrightsquigarrow$ PD problem for $(\widetilde{X}, \widetilde{\Gamma})$.

- W.l.o.g. $\Gamma = (\Gamma_1, ..., \Gamma_k)$ with $\ell(\Gamma_i) \leq 2\pi$.
- $\gamma_i : \mathbb{S}^1 \to \Gamma_i$ constant speed parametrization.
- For $i = 1, ..., k$ set $Z_i := \mathbb{S}^1 \times [0, 1]$.

The construction

- W.l.o.g. $\Gamma = (\Gamma_1, ..., \Gamma_k)$ with $\ell(\Gamma_i) \leq 2\pi$ for all i.
- $\gamma_i : \mathbb{S}^1 \to \Gamma_i$ constant speed parametrization.
- For $i = 1, ..., k$ set $Z_i := \mathbb{S}^1 \times [0, 1]$.

- Let
$$\widetilde{X} := (X \sqcup Z_1 \sqcup ... \sqcup Z_k) / \sim$$
where $\gamma_i(t) \sim (t, 0)$ for $(t, 0) \in Z_i$.

The construction

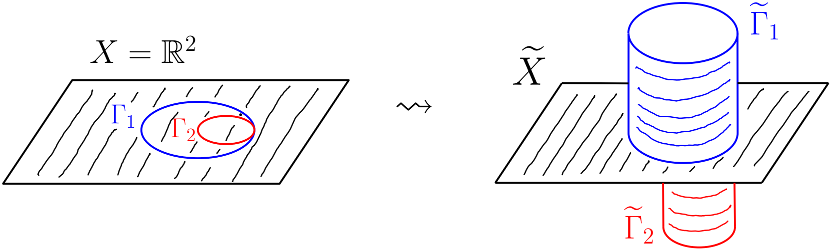
- W.l.o.g. $\Gamma = (\Gamma_1, ..., \Gamma_k)$ with $\ell(\Gamma_i) \leq 2\pi$ for all i.
- $\gamma_i : \mathbb{S}^1 \to \Gamma_i$ constant speed parametrization.
- For $i = 1, ..., k$ set $Z_i := \mathbb{S}^1 \times [0, 1]$.

- Let
$$\widetilde{X} := (X \sqcup Z_1 \sqcup ... \sqcup Z_k) \,/ \sim$$
where $\gamma_i(t) \sim (t, 0)$ for $(t, 0) \in Z_i$.

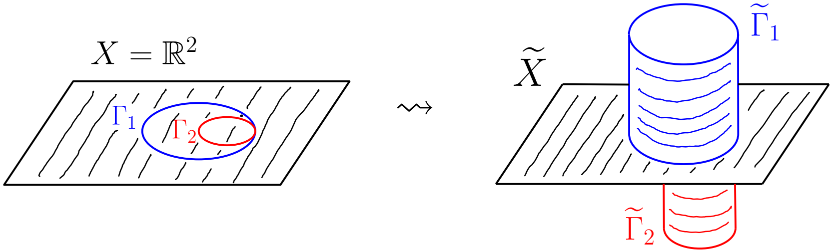
- Let
$$\widetilde{\Gamma}_i := \mathbb{S}^1 \times \{1\} \subset Z_i \subset \widetilde{X}$$
and $\widetilde{\Gamma} := (\widetilde{\Gamma}_1, ..., \widetilde{\Gamma}_k)$.

The construction

- Let
$$\widetilde{X} := (X \sqcup Z_1 \sqcup \ldots \sqcup Z_k) \,/\, \sim$$
where $\gamma_i(t) \sim (t, 0)$ for $(t, 0) \in Z_i$.
- Let
$$\widetilde{\Gamma}_i := \mathbb{S}^1 \times \{1\} \subset Z_i \subset \widetilde{X}$$
and $\widetilde{\Gamma} := (\widetilde{\Gamma}_1, \ldots, \widetilde{\Gamma}_k)$.

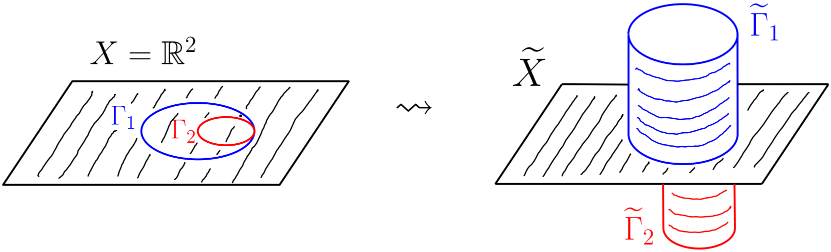


Sketch of the argument



- $\widetilde{\Gamma} \subset \widetilde{X}$ configuration of disjoint Jordan curves.
- $X \subset \widetilde{X}$ isometrically and $\exists$ 1-Lipschitz projection $P : \widetilde{X} \to X$.

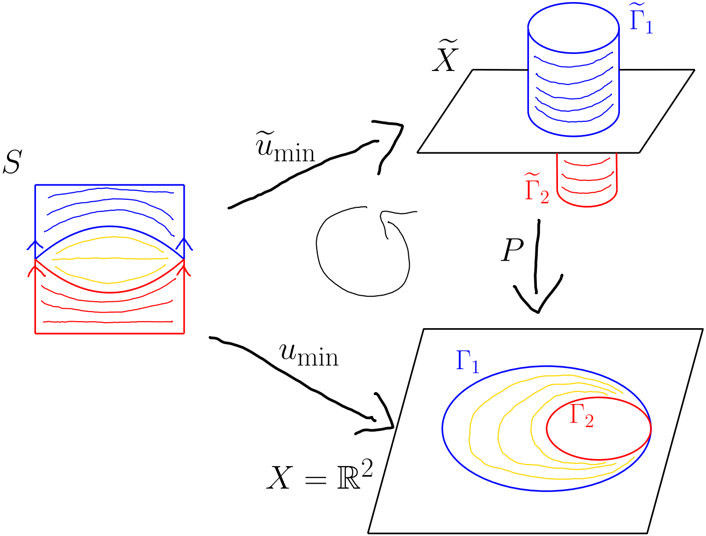
Sketch of the argument



- $\widetilde{\Gamma} \subset \widetilde{X}$ configuration of disjoint Jordan curves.
- $X \subset \widetilde{X}$ isometrically and $\exists$ 1-Lipschitz projection $P : \widetilde{X} \to X$.
- By Theorem 2 exists area minimizer $\widetilde{u}_{\text{min}} \in \Lambda(S, \widetilde{\Gamma}, \widetilde{X})$

$\rightsquigarrow$ $u_{\text{min}} := P \circ \widetilde{u}_{\text{min}} \in \Lambda(S, \Gamma, X)$ is an area minimizer $\square$

Sketch of the argument



Thank You!