Area minimizing surfaces for singular configurations of boundary curves

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(joint work with Martin Fitzi)

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1. Classical Plateau-Douglas problem

Standing assumptions

- Fix a 2-dimensional differentiable manifold S which is...
  - compact,
  - connected,
  - orientable,
  - and has boundary.

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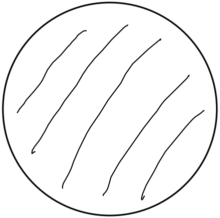
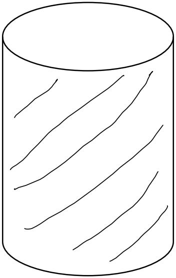
  - compact,
  - connected,
  - orientable,
  - and has boundary.
- S uniquely determined by...
  - $k :=$ number of boundary components, and
  - $p :=$ genus of S.

Standing assumptions

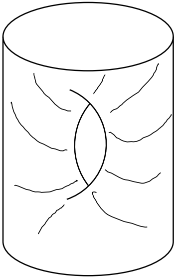
- S compact, connected, orientable surface.
- $k = \sharp$ boundary components, $\quad p =$ genus.



$k = 1 \quad p = 0$



$k = 2 \quad p = 0$



$k = 2 \quad p = 1$

Standing assumptions

- S compact, connected, orientable surface with k boundary components.
- $\Gamma \subset \mathbb{R}^n$ a configuration of k disjoint Jordan curves of finite length.

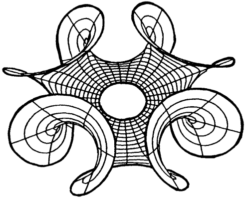
Motivation

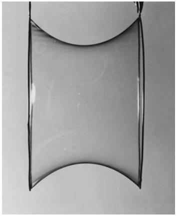
- S compact, connected, orientable surface with k boundary components.
- $\Gamma \subset \mathbb{R}^n$ a configuration of k disjoint Jordan curves of finite length.

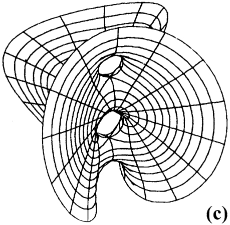
Motivating question

Is there a minimal surface $T_{\min} \subset \mathbb{R}^n$ s.t.

- $\partial T_{\min} = \Gamma$ and
- $T_{\min} \simeq S$?







(c)

Figures taken from Dierkes-Hildebrandt-Sauvigny "Minimal surfaces"

The area functional

- Sobolev space $W^{1,2}(S, \mathbb{R}^n)$ defined by

$$u \in W^{1,2}(S, \mathbb{R}^n) :\Leftrightarrow u \in L^2 \wedge du \in L^2.$$

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- By the area formula

$$\mathrm{Area}(u) = \int_{\mathbb{R}^n} \mathrm{card}\{u^{-1}(y)\} \, \mathrm{d}\mathcal{H}^2(y).$$

Plateau-Douglas problem

Motivating Question: Is there a minimal surface $T_{\min} \subset \mathbb{R}^n$ s.t.

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- $u_{\min} \in \Lambda(M, \Gamma, \mathbb{R}^n)$ is called *area minimizer* if

$$\text{Area}(u_{\min}) \leq \text{Area}(u) \qquad ; \forall u \in \Lambda(S, \Gamma, \mathbb{R}^n).$$

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Question (Plateau-Douglas problem)

- *Is there an area minimizer $u_{\text{min}} \in \Lambda(S, \Gamma, \mathbb{R}^n)$?*
- *If yes: Regularity of u_{min}?*

Existence & Regularity of area minimizers

Set $\Lambda(S, \Gamma, \mathbb{R}^n) := \{u \in W^{1,2}(S, \mathbb{R}^n) : u|_{\partial S} \text{ parametrizes } \Gamma\}$.

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- Existence: **Yes!**

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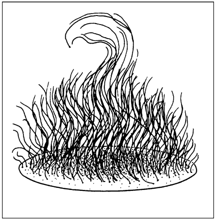
- Is there an area minimizer $u_{\min} \in \Lambda(S, \Gamma, \mathbb{R}^n)$?
- If yes: Regularity of $u_{\min}$?

- Existence: **Yes!**
- However: In general no reasonable regularity!



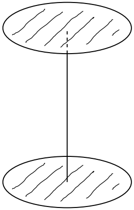


Figure taken from Dierkes-Hildebrandt-Sauvigny "Minimal surfaces"

Energy functional

- Set
$$\mathcal{R}(S) := \{\text{smooth Riemannian metrics on } S\}.$$
- For $g \in \mathcal{R}(S)$ and $u \in W^{1,2}(S, \mathbb{R}^n)$ define *Dirichlet energy*
$$E^2(u, g) := \frac{1}{2} \cdot \int_S |du|_g^2 \ \mathrm{d}A_g.$$

Energy vs. Area

Let $u \in W^{1,2}(S, \mathbb{R}^n)$.

- $\text{Area}(u) = \inf_{g \in \mathcal{R}(S)} E^2(u, g)$, (Fitzi-Wenger, 2019).

Energy vs. Area

Let $u \in W^{1,2}(S, \mathbb{R}^n)$.

- $\text{Area}(u) = \inf_{g \in \mathcal{R}(S)} E^2(u, g)$, (Fitzi-Wenger, 2019).
- The following are equivalent:
  - $\text{Area}(u) = E^2(u, g)$.
  - $u : (S, g) \to \mathbb{R}^n$ is weakly conformal.
  - $E^2(u, g) \leq E^2(u, h)$ for all $h \in \mathcal{R}(S)$.

Energy minimizers

- $\text{Area}(u) = \inf_{g \in \mathcal{R}(S)} E^2(u, g)$

- $(u_{\min}, g_{\min})$ where $u_{\min} \in \Lambda(S, \Gamma, \mathbb{R}^n)$ and $g_{\min} \in \mathcal{R}(S)$ is called *energy minimizing* pair if

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$(u_{\min}, g_{\min})$ energy minimizing $\rightsquigarrow$ **$u_{\min}$ area minimizer**

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$(u_{\min}, g_{\min})$ **energy minimizing** $\rightsquigarrow$ $u_{\min}$ **area minimizer**

Question

- *Are there* $(u_{\min}, g_{\min})$ *energy minimizing?*
- *If yes: Regularity of* $u_{\min}$*?*

Regularity of energy minimizers

- Are there $(u_{\min}, g_{\min})$ energy minimizing?
- If yes: Regularity of $u_{\min}$?

If $(u_{\min}, g_{\min})$ is energy minimizing

$\rightsquigarrow$ $u_{\min} : (S, g_{\min}) \to \mathbb{R}^n$ is...

- weakly conformal, and
- a branched minimal surface.

Existence of energy minimizers

Are there $(u_{\min}, g_{\min})$ energy minimizing?

Existence of energy minimizers

Are there $(u_{\text{min}}, g_{\text{min}})$ energy minimizing?

In general: No!

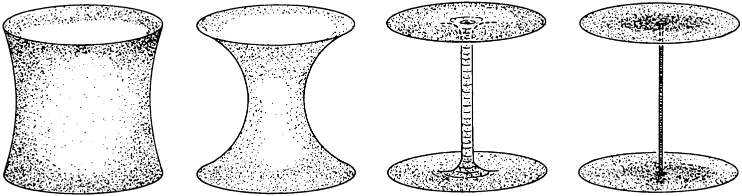

Figure taken from Dierkes-Hildebrandt-Tromba "Global analysis of minimal surfaces"

Existence of energy minimizers

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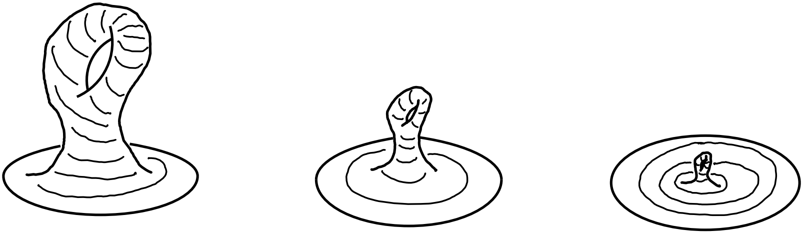

Reductions

S^* orientable compact surface (possibly disconnected!).

- S^* is called a *reduction* of S if
  - S^* has k boundary components and,
  - $\text{genus}(S^*) := p^* \leq p$.

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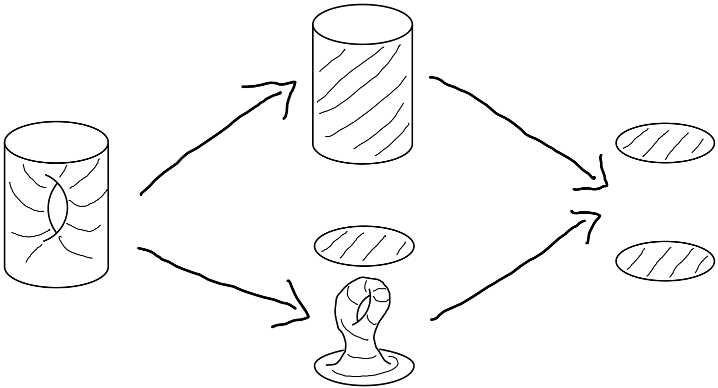

Douglas condition

S^* is called a *reduction of S* if

- S^* has k boundary components and,
- $\mathrm{genus}(S^*) := p^* \leq p$.

The *Douglas condition* holds if

$$\inf_{u \in \Lambda(S,\Gamma,\mathbb{R}^n)} \mathrm{Area}(u) < \inf_{u^* \in \Lambda(S^*,\Gamma,\mathbb{R}^n)} \mathrm{Area}(u^*)$$

for every reduction $S^* \neq S$.

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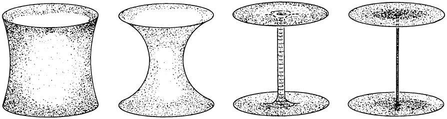
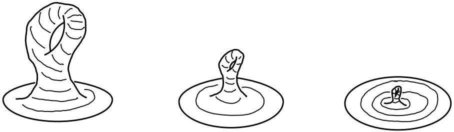


$\rightsquigarrow$ Douglas condition violated for $S^* = \mathbb{D} \sqcup \mathbb{D}$.



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Douglas result

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for every reduction S^* of S.

Theorem (Douglas '39)

If Douglas condition holds $\rightsquigarrow \exists(u_{\min}, g_{\min})$ *energy minimizing.*

Douglas result

Theorem (Douglas '39)

If Douglas condition holds $\rightsquigarrow \exists (u_{\mathrm{min}}, g_{\mathrm{min}})$ *energy minimizing.*

Corollary

If Douglas condition holds $\rightsquigarrow \exists$ *area minimizer* u_{min} *s.t.*

- u_{min} *is a branched minimal surface, and*
- $\exists g \in \mathcal{R}(S)$ *s.t.* $u_{\mathrm{min}} : (S, g) \to \mathbb{R}^n$ *is weakly conformal.*

2. Plateau-Douglas problem for singular boundary values

Standing assumptions

Classical problem:

- S compact, connected, orientable surface with k boundary components.
- $\Gamma \subset \mathbb{R}^n$ a configuration of k disjoint Jordan curves of finite length.

Standing assumptions

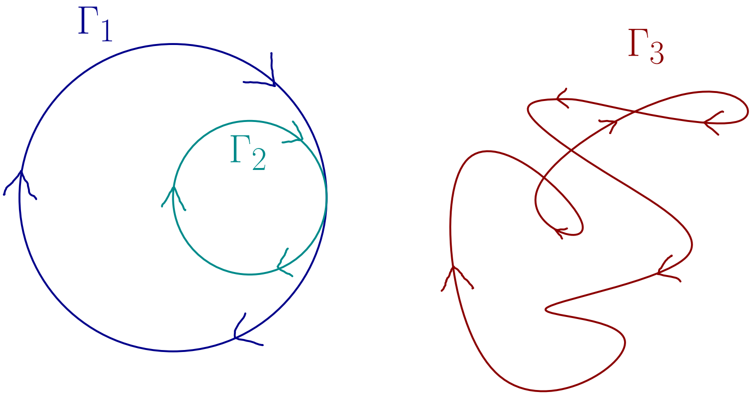
Now:

- S compact, connected, orientable surface with k boundary components.
- $\Gamma \subset \mathbb{R}^n$ a configuration of k ~~disjoint Jordan~~ curves of finite length.

Standing assumptions

- S compact, connected, orientable surface with k boundary components.
- $\Gamma \subset \mathbb{R}^n$ a configuration of k closed curves of finite length.

$$\Gamma = (\Gamma_1, \Gamma_2, \Gamma_3)$$



Plateau-Douglas problem for singular boundary values

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Question (PD problem for singular boundary values)

If Douglas-condition holds

$\rightsquigarrow \exists$ *'reasonably regular' area minimizer* $u_{\text{min}} \in \Lambda(S, \Gamma, \mathbb{R}^n)$*?*

Existence of energy minimizers?

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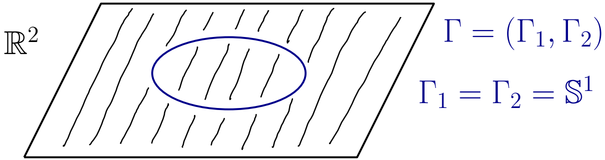

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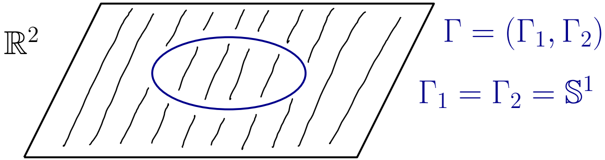

$$S = \mathbb{S}^1 \times [0,1] \rightsquigarrow \inf_{u \in \Lambda(S,\Gamma,X)} \mathrm{Area}(u) = 0$$

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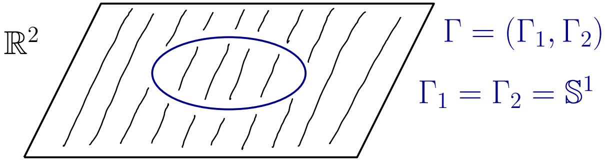

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$$S^* \text{ reduction} \rightsquigarrow S^* = \mathbb{D} \sqcup \mathbb{D} \rightsquigarrow \inf_{u \in \Lambda(S^*,\Gamma,X)} \text{Area}(u^*) = 2 \cdot \pi$$

Existence of energy minimizers?

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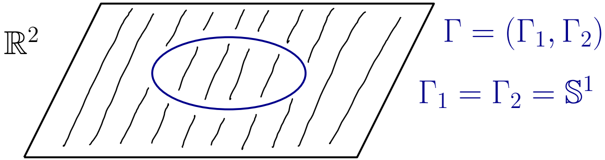

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If $(u_{\min}, g_{\min})$ energy minimizing for (S, Γ)

$\rightsquigarrow u_{\min}$ weakly conformal $+ \mathrm{Area}(u_{\min}) = 0$

$\rightsquigarrow u_{\min}$ constant $\lightning$

Previous results

Question (PD problem for singular boundary values)

If Douglas-condition holds
$\rightsquigarrow \exists$ *'reasonably regular' area minimizer* $u_{\min} \in \Lambda(S, \Gamma, \mathbb{R}^n)$*?*

- "Yes" for $S = \mathbb{D}$ (Hass '91):
  - $u_{\min}$ is continuous.
  - $u_{\min}$ is a branched minimal surface on $S \setminus u^{-1}(\Gamma)$.
  - Caution: In general $u \notin \Lambda(S, \Gamma, \mathbb{R}^n)$ and no weak conformality.

Previous results

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  - Caution: In general $u \notin \Lambda(S, \Gamma, \mathbb{R}^n)$ and no weak conformality.
- Additionally assuming "Condition of adhesion"

  $\rightsquigarrow$ existence of energy minimizing $(u_{\min}, g_{\min})$ (Iseri '96).

Main result

Question (PD problem for singular boundary values)

If Douglas-condition holds
$\rightsquigarrow \exists$ *'reasonably regular' area minimizer* $u_{\text{min}} \in \Lambda(S, \Gamma, \mathbb{R}^n)$*?*

Theorem (C.-Fitzi, 2020)

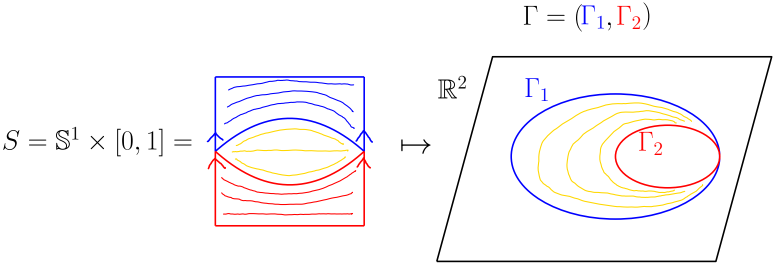
Yes!

- u_{min} *is Hoelder continuous.*
- *There is* $g \in \mathcal{R}(S)$ *such that* $u_{\text{min}} : (S, g) \to \mathbb{R}^n$ *is a* ***weakly conformal*** *branched minimal surface on* $S \setminus u^{-1}(\Gamma)$*.*
- *If* Γ *is* $C^2 \rightsquigarrow u_{\text{min}}$ *locally Lipschitz on* $S \setminus \partial S$*.*

Main result

If Douglas-condition holds $\rightsquigarrow$ $\exists$ area minimizer $u_{\text{min}} \in \Lambda(S, \Gamma, X)$ s.t.

- u_{min} is Hoelder continuous.
- There is $g \in \mathcal{R}(S)$ such that $u_{\text{min}} : (S, g) \to \mathbb{R}^n$ is a **weakly conformal** branched minimal surface on $S \setminus u^{-1}(\Gamma)$.
- If Γ is C^2 $\rightsquigarrow$ u_{min} locally Lipschitz on $S \setminus \partial S$.



3. Plateau-Douglas problem in singular spaces

Standing assumptions

Classical problem:

- S compact, connected, orientable surface with k boundary components.
- $\Gamma \subset \mathbb{R}^n$ a configuration of k disjoint Jordan curves of finite length.

Standing assumptions

Now:

- S compact, connected, orientable surface with k boundary components.
- $X = (X, d)$ a proper metric space.
- $\Gamma \subset X$ a configuration of k disjoint Jordan curves of finite length.

Geometric analysis in metric spaces

- S compact, connected, orientable surface with k boundary components.
- $X = (X, d)$ a proper metric space.
- $\Gamma \subset X$ a configuration of k disjoint Jordan curves of finite length.

- $u \in W^{1,2}(S, X)$ if
  - for all $x \in X$ one has $u_x := d(x, u(-)) \in W^{1,2}(S, \mathbb{R})$, and
  - for fixed $g \in \mathcal{R}(S)$ there is $h \in L^2(S)$ s.t

$$|du_x|_g \leq h$$

    a.e. independently of x.

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$$|du_x|_g \leq h$$

    a.e. independently of x.

- Trace $u|_{\partial S}$ defined for $u \in W^{1,2}(S, X)$.

$\rightsquigarrow$ $\Lambda(S, \Gamma, X) := \{u \in W^{1,2}(S, X) : \ u|_{\partial S} \text{ parametrizes } \Gamma\}$

Geometric analysis in metric spaces

- For $u \in W^{1,2}(S,X)$ define
$$\mathrm{Area}(u) := \int_X \mathrm{card}\{u^{-1}(y)\}\, \mathrm{d}\mathcal{H}^2(y).$$
- For $u \in W^{1,2}(S,X)$ and $g \in \mathcal{R}(S)$ define
$$E_+^2(u,g) := \inf_h \int_S h^2 \; dA_g.$$

Plateau-Douglas problem in metric spaces

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Question (PD problem for singular spaces)

If Douglas-condition holds

$\rightsquigarrow \exists$ *'reasonably regular' area minimizer* $u_{\min} \in \Lambda(S, \Gamma, X)$*?*

Energy minimizers are not(!) area minimizers

- ~~$\text{Area}(u) = \inf_{g \in \mathcal{R}(S)} E_+^2(u, g)$~~

- $(u_{\min}, g_{\min})$ where $u_{\min} \in \Lambda(S, \Gamma, X)$ and $g_{\min} \in \mathcal{R}(S)$ is called *energy minimizing* pair if

$$E_+^2(u_{\min}, g_{\min}) \leq E_+^2(u, g) \quad ; \forall u \in \Lambda(S, \Gamma, X); \forall g \in \mathcal{R}(S).$$

$(u_{\min}, g_{\min})$ energy minimizing $\not\rightsquigarrow$ **$u_{\min}$ area minimizer**

Replacement for conformality

- For $u \in W^{1,2}(S, \mathbb{R}^n)$ the following are equivalent:
  - $\text{Area}(u) = E_+^2(u, g)$.
  - $u : (S, g) \to \mathbb{R}^n$ is weakly conformal.
  - $E_+^2(u, g) \leq E_+^2(u, h)$ for all $h \in \mathcal{R}(S)$.

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- We say that $u : (S, g) \to X$ is *infinitesimally isotropic* if

$$E_+^2(u, g) \leq E_+^2(u, h) \qquad ; \forall h \in \mathcal{R}(S).$$

Plateau-Douglas problem in metric spaces

Theorem (C.-Fitzi, 2020)

If Douglas-condition holds, then

- $\exists$ *area minimizer* $u_{\min} \in \Lambda(S, \Gamma, X)$, *and*
- $\exists g \in \mathcal{R}(S)$ *s.t.* $u_{\min} : (S, g) \to X$ *is infinitesimally isotropic.*

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- Previous results:
  - The case $S = \mathbb{D}$ (Lytchak-Wenger 2016)
  - For spaces X which satisfy a local quadratic isoperimetric inequality (Fitzi-Wenger 2020)
- New e.g. for general complete Riemannian manifolds X.

4. Proof sketch

Statements

- $\Gamma \subset \mathbb{R}^n$ a configuration of k closed curves.

Theorem 1

If Douglas-condition holds

$\rightsquigarrow \exists$ 'reasonable' area minimizer $u_{\min} \in \Lambda(S, \Gamma, \mathbb{R}^n)$

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Theorem 1

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- X a proper metric space,
- $\Gamma \subset X$ a configuration of k disjoint Jordan curves

Theorem 2

If Douglas-condition holds

$\rightsquigarrow \exists$ 'reasonable' area minimizer $u_{\min} \in \Lambda(S, \Gamma, X)$

Proof of Theorem 1 (modulo Theorem 2)

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$$\text{PD problem for } (X, \Gamma) \leftrightsquigarrow \text{PD problem for } (\widetilde{X}, \widetilde{\Gamma}).$$

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- Apply Theorem 2 to solve PD problem for $(\widetilde{X}, \widetilde{\Gamma})$. $\square$

The construction

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- W.l.o.g. $\Gamma = (\Gamma_1, \ldots, \Gamma_k)$ with $\ell(\Gamma_i) \leq 2\pi$.

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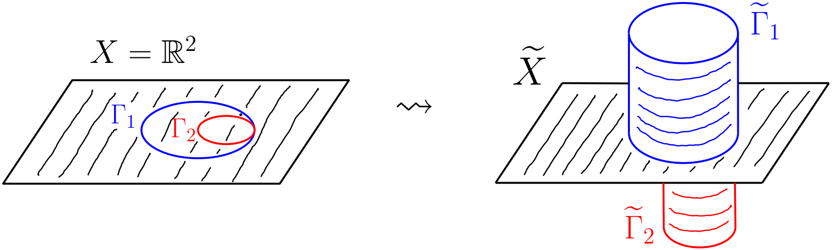
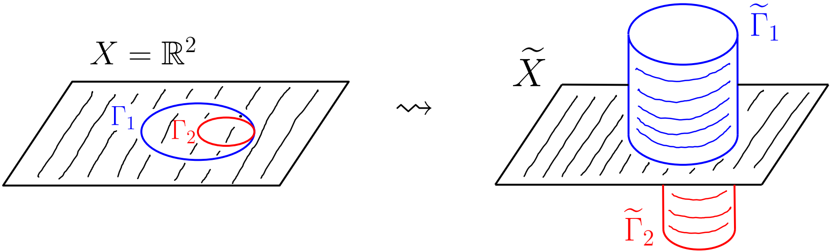


Sketch of the argument



- $\widetilde{\Gamma} \subset \widetilde{X}$ configuration of disjoint Jordan curves of finite length.
- $X \subset \widetilde{X}$ isometrically and $\exists$ 1-Lipschitz projection $P : \widetilde{X} \to X$.

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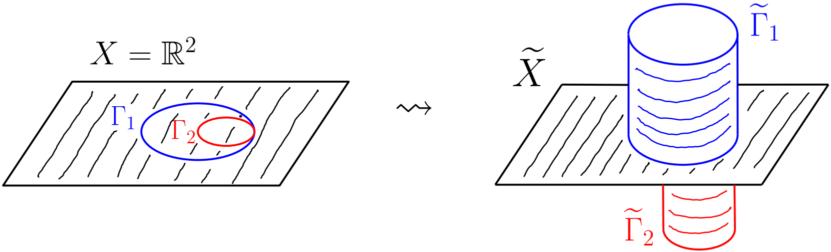

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$$\text{Area}(P \circ \widetilde{u}) \leq \text{Area}(\widetilde{u}) - 2\pi \cdot k.$$

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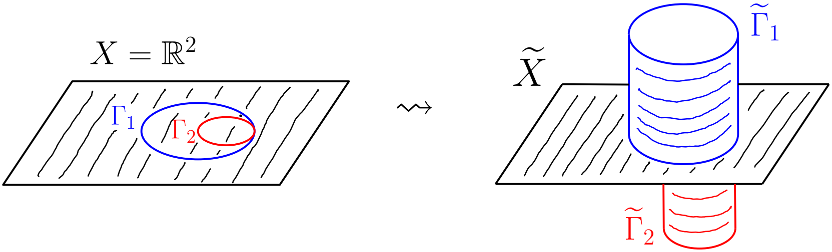

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- For any $u \in \Lambda(S, \Gamma, X)$ and $\varepsilon > 0$ there is $\widetilde{u} \in \Lambda(S, \widetilde{\Gamma}, \widetilde{X})$ s.t.
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- $$\inf_{\widetilde{u} \in \Lambda(S, \widetilde{\Gamma}, \widetilde{X})} \text{Area}(\widetilde{u}) = \inf_{u \in \Lambda(S, \Gamma, X)} \text{Area}(u) + 2\pi \cdot k.$$

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- $\widetilde{\Gamma} \subset \widetilde{X}$ configuration of disjoint Jordan curves of finite length.
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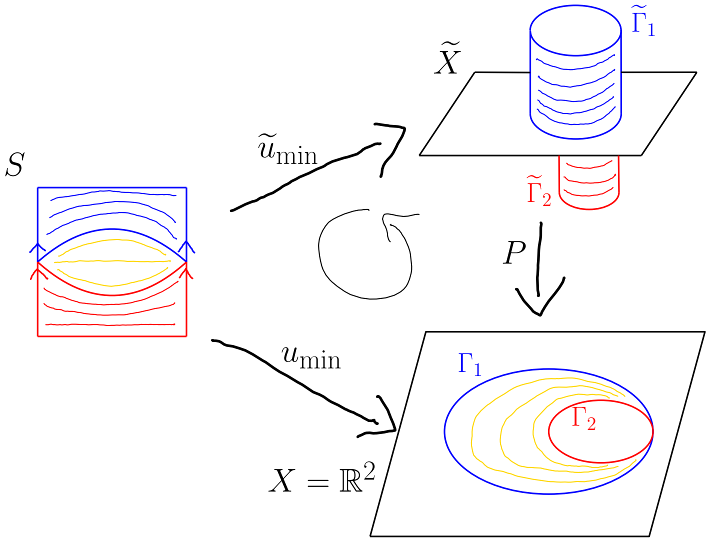
- Douglas condition holds for $(S, \widetilde{\Gamma}, \widetilde{X})$

- By Theorem 2 exists area minimizer $\widetilde{u}_{\text{min}} \in \Lambda(S, \widetilde{\Gamma}, \widetilde{X})$

$\rightsquigarrow$ $u_{\text{min}} := P \circ \widetilde{u}_{\text{min}} \in \Lambda(S, \Gamma, X)$ is an area minimizer □

Sketch of the argument



Sketch of Theorem 2

- There are proper metric spaces $(X_n)_{n\in\mathbb{N}}$ such that
  - X_n satisfies a local quadratic isoperimetric inequality,
  - $X \subset X_n$ isometrically, and
  - $d_H(X_n, X) \to 0$.
- Apply Fitzi-Wenger to obtain u_n solution to the PD problem for (Γ, X_n)

$\leadsto$ $u := \lim_{n\to\infty} u_n$ is solution to PD problem (Γ, X) $\square$

Thank You!