Area minimizing surfaces for singular boundary values

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1. Classical Plateau-Douglas problem

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- S is uniquely determined by...
  - k := number of boundary components, and
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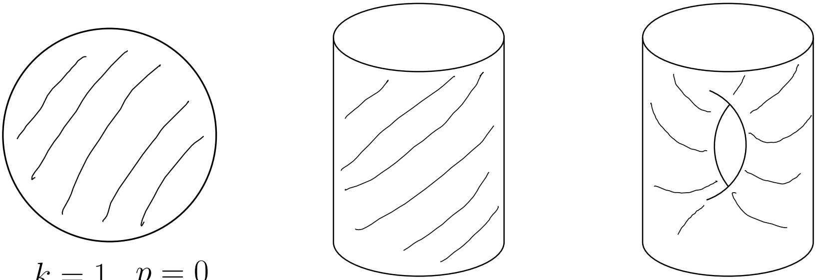

$k = 1 \quad p = 0$ $\qquad$ $k = 2 \quad p = 0$ $\qquad$ $k = 2 \quad p = 1$

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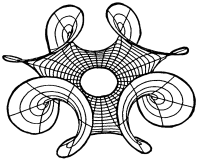
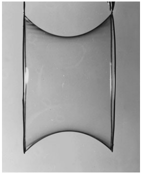
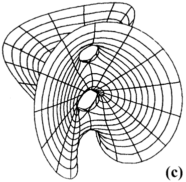
Motivating question

Is there a minimal surface $T_{\text{min}} \subset \mathbb{R}^n$ s.t.

- $\partial T_{\text{min}} = \Gamma$, and
- $T_{\text{min}} \simeq S$?







Figures taken from Dierkes-Hildebrandt-Sauvigny "Minimal surfaces"

Plateau-Douglas problem

- Sobolev space $W^{1,2}(S, \mathbb{R}^n)$ defined by

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- $u_{\min} \in \Lambda(S,\Gamma,\mathbb{R}^n)$ is called *area minimizer* if

$$\text{Area}(u_{\min}) = \inf_{u \in \Lambda(S,\Gamma,\mathbb{R}^n)} \text{Area}(u).$$

Question (Plateau-Douglas problem)

- Is there an area minimizer $u_{\min} \in \Lambda(S,\Gamma,\mathbb{R}^n)$?
- If yes: Regularity of $u_{\min}$?

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- Existence: **Yes!**

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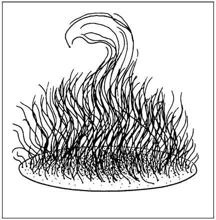
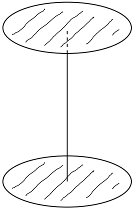
- However: In general no reasonable regularity!





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Energy minimization

- Set
$$\mathcal{R}(S) := \{\text{smooth Riemannian metrics on } S\}.$$
- For $g \in \mathcal{R}(S)$ and $u \in W^{1,2}(S, \mathbb{R}^n)$ define *Dirichlet energy*
$$E^2(u, g) := \frac{1}{2} \cdot \int_S |du|_g^2 \ \mathrm{d}A_g.$$

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- So $(u_{\min}, g_{\min})$ energy minimzing pair

$\rightsquigarrow$ $u_{\min} : (S, g_{\min}) \to \mathbb{R}^n$ is harmonic and weakly conformal.

Existence of energy minimizers

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In general: No!

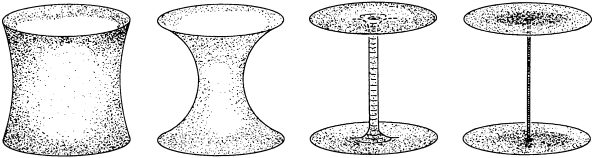
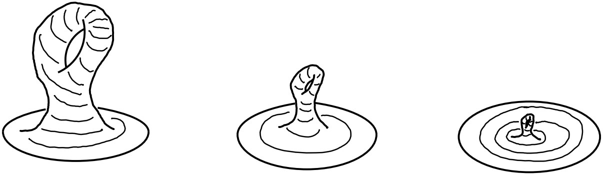


Figure taken from Dierkes-Hildebrandt-Tromba "Global analysis of minimal surfaces"



Douglas result

- An orientable compact surface S^* is called a *reduction* of S if
  - S^* has k boundary components, and
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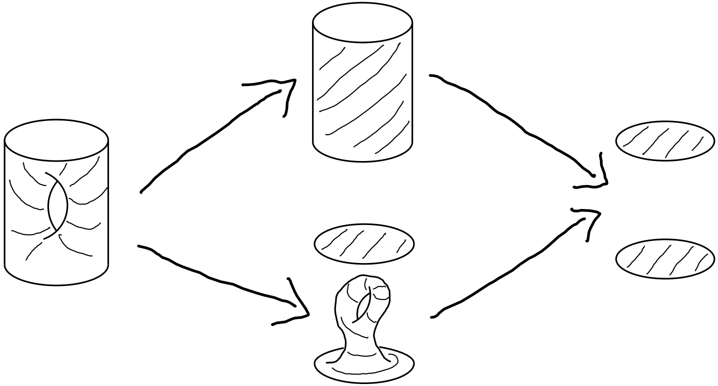

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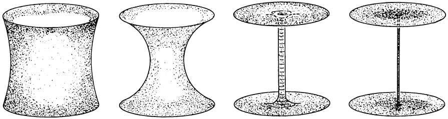
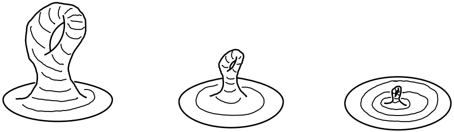


$\rightsquigarrow$ Douglas condition violated for $S^* = \mathbb{D} \sqcup \mathbb{D}$.



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Theorem (Douglas '39)

If Douglas condition holds $\rightsquigarrow \exists (u_{\min}, g_{\min})$ energy minimizing.

Corollary

If Douglas condition holds

$\rightsquigarrow$ $\exists$ area minimizer $u_{\min} \in \Lambda(S, \Gamma, \mathbb{R}^n)$, and

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2. Plateau-Douglas problem for singular boundary values

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Classical problem:

- S compact, connected, orientable surface with k boundary components.
- $\Gamma \subset \mathbb{R}^n$ a configuration of k disjoint Jordan curves of finite length.

Plateau-Douglas problem for singular boundary values

Now:

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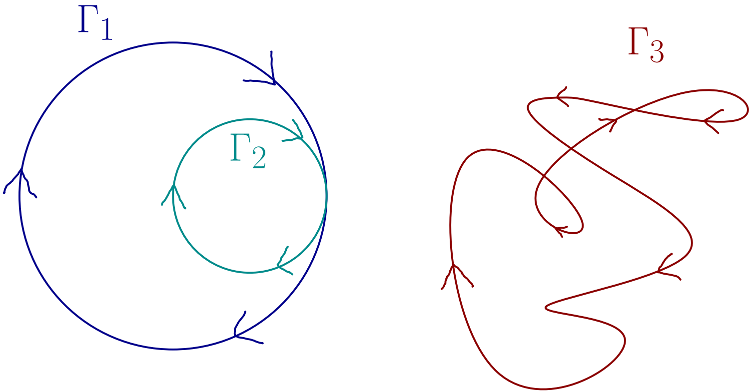
$$\Gamma = (\Gamma_1, \Gamma_2, \Gamma_3)$$



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$\rightsquigarrow$ $\exists$ 'reasonable' area minimizer $u_{\min} \in \Lambda(S, \Gamma, \mathbb{R}^n)$?

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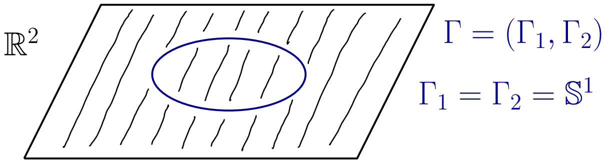

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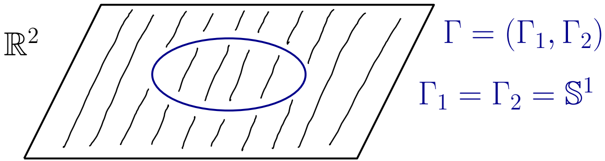

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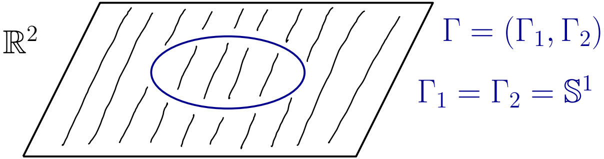

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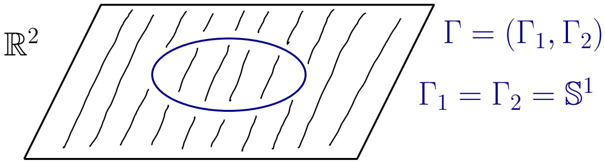

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- If $(u_{\min}, g_{\min})$ energy minimizing for (S, Γ)

$\rightsquigarrow$ $u_{\min}$ weakly conformal $+ \text{Area}(u_{\min}) = 0$

$\rightsquigarrow$ $u_{\min}$ constant $\lightning$

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$\rightsquigarrow$ $\exists$ 'reasonable' area minimizer $u_{\min} \in \Lambda(S, \Gamma, \mathbb{R}^n)$?

Hass ('91):

- "Yes" for $S = \mathbb{D}$.
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Theorem 1 (C.-Fitzi, 2020)

Yes, there is an area minimizing $u_{\min} \in \Lambda(S, \Gamma, \mathbb{R}^n)$. Furthermore:

- $u_{\min}$ is Hoelder continuous.
- There is $g \in \mathcal{R}(S)$ such that $u_{\min} : (S, g) \to \mathbb{R}^n$ is harmonic and weakly conformal on $S \setminus u^{-1}(\Gamma)$.
- If Γ is C^2 $\rightsquigarrow$ $u_{\min}$ locally Lipschitz on $S \setminus \partial S$.

3. Plateau-Douglas problem in singular spaces

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- $u_{\min} \in \Lambda(M, \Gamma, X)$ is called *area minimizer* if

$$\text{Area}(u_{\min}) = \inf_{u \in \Lambda(S\Gamma, X)} \text{Area}(u).$$

Question

If Douglas-condition holds

$\rightsquigarrow$ $\exists$ 'reasonable' area minimizer $u_{\min} \in \Lambda(S, \Gamma, X)$?

Energy in the metric setting

- $(u_{\min}, g_{\min}) \in \Lambda(S, \Gamma, X) \times \mathcal{R}(S)$ is called *energy minimizing* pair if

$$E_+^2(u_{\min}, g_{\min}) \leq E_+^2(u, g) \qquad ; \forall u \in \Lambda(S, \Gamma, X); \forall g \in \mathcal{R}(S).$$

- For $u \in W^{1,2}(S, X)$ in general

$$\mathrm{Area}(u) \neq \inf_{g \in \mathcal{R}(S)} E_+^2(u, g).$$

$(u_{\min}, g_{\min})$ **energy minimizing** $\quad \not\rightsquigarrow \quad$ $u_{\min}$ **area minimizer**

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$(u_{\min}, g_{\min})$ energy minimizing $\not\rightsquigarrow$ **$u_{\min}$ area minimizer**

- Let $u \in W^{1,2}(S, \mathbb{R}^n)$ and $g \in \mathcal{R}(S)$. TFAE:
  - $E_+^2(u, g) = \inf_{h \in \mathcal{R}(S)} E_+^2(u, h)$.
  - $u \colon (S, g) \to \mathbb{R}^n$ is weakly conformal.
- We say that $u \colon (S, g) \to X$ is *infinitesimally isotropic* if
$$E_+^2(u, g) = \inf_{h \in \mathcal{R}(S)} E_+^2(u, h).$$

Plateau-Douglas problem in singular spaces

- S compact, connected, orientable surface with k boundary components.
- $X = (X, d)$ a proper metric space.
- $\Gamma \subset X$ a configuration of k disjoint Jordan curves of finite length.

Theorem 2 (C.-Fitzi, 2020)

If Douglas-condition holds, then

$\rightsquigarrow$ $\exists$ area minimizer $u_{\min} \in \Lambda(S, \Gamma, X)$, and

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- Previous results:
  - The case $S = \mathbb{D}$ (Lytchak-Wenger 2016)
  - For spaces X which satisfy a local quadratic isoperimetric inequality (Fitzi-Wenger 2020)
- New e.g. for general complete Riemannian manifolds X.

4. Proof sketch

Idea

- $\Gamma \subset \mathbb{R}^n$ a configuration of closed curves.

Theorem 1 (C.-Fitzi)

If Douglas-condition holds $\rightsquigarrow \exists$ 'reasonable' area minimizer $u_{\text{min}} \in \Lambda(S, \Gamma, \mathbb{R}^n)$.

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- Given $Y = \mathbb{R}^n$ regular space and $\Gamma \subset Y$ singular configuration.
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- Apply Theorem 2 to solve PD problem for $(\widetilde{Y}, \widetilde{\Gamma})$. $\square$

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- W.l.o.g. $\Gamma = (\Gamma_1, ..., \Gamma_k)$ with $\ell(\Gamma_i) \leq 2\pi$.

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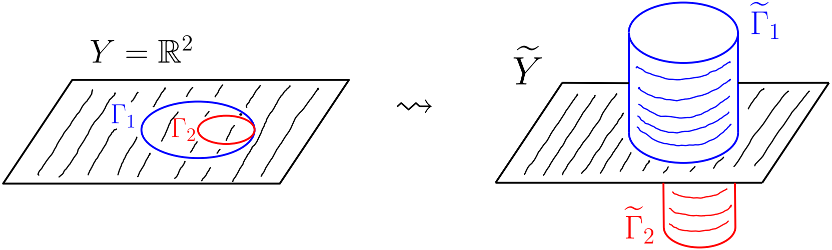
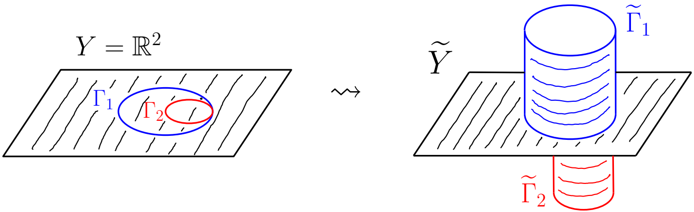


The proof



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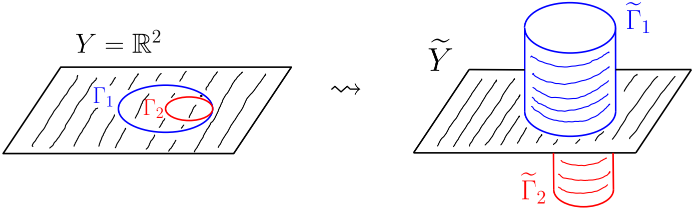

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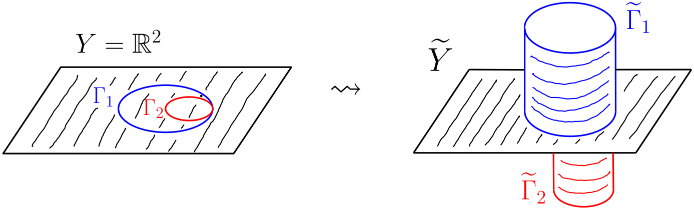

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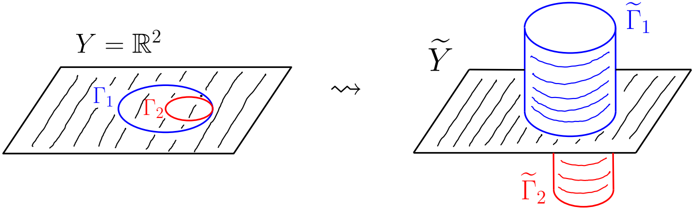

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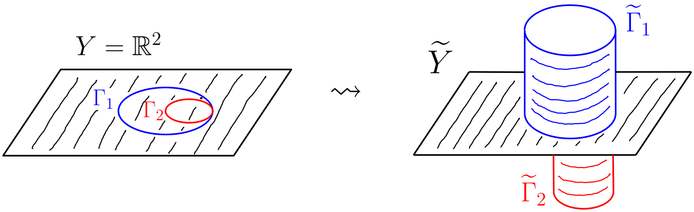

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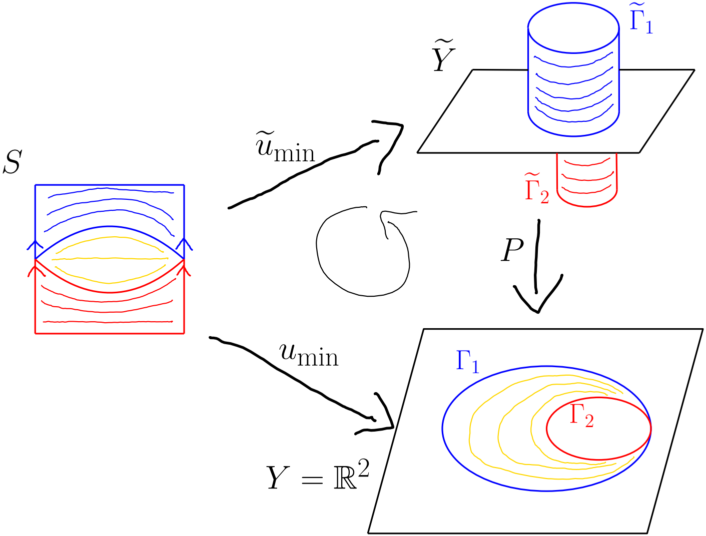

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$\rightsquigarrow$ $u := \lim_{n\to\infty} u_n$ is solution to PD problem (Γ, X) $\square$

Thank You!