

Area minimizing surfaces for singular boundary values

Paul Creutz

(joint work with Martin Fitzi)

University of Cologne

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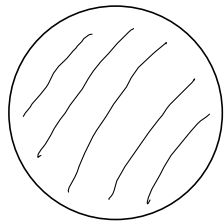
1. Classical Plateau-Douglas problem

Motivation

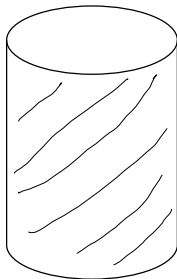
- ▶ Fix a 2-dimensional differentiable manifold S which is compact, connected, orientable and has boundary.

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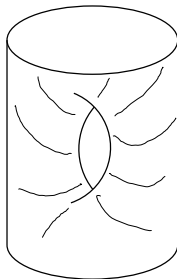
- ▶ Fix a 2-dimensional differentiable manifold S which is compact, connected, orientable and has boundary.
- ▶ S is uniquely determined by...
 - $k :=$ number of boundary components, and
 - $p :=$ genus of S .



$$k = 1 \quad p = 0$$



$$k = 2 \quad p = 0$$



$$k = 2 \quad p = 1$$

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- ▶ Let $\Gamma \subset \mathbb{R}^n$ be a configuration of k disjoint Jordan curves of finite length.

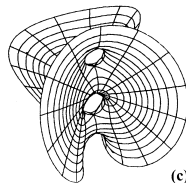
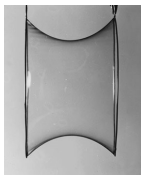
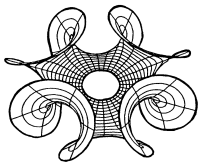
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Motivating question

Is there a minimal surface $T_{\min} \subset \mathbb{R}^n$ s.t.

- $\partial T_{\min} = \Gamma$ and
- $T_{\min} \simeq S$?



(c)

Plateau-Douglas problem

- ▶ Sobolev space $W^{1,2}(S, \mathbb{R}^n)$ defined by

$$u \in W^{1,2}(S, \mathbb{R}^n) :\Leftrightarrow u \in L^2 \wedge du \in L^2.$$

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- ▶ $u_{\min} \in \Lambda(S, \Gamma, \mathbb{R}^n)$ is called *area minimizer* if

$$\text{Area}(u_{\min}) \leq \text{Area}(u) \quad ; \forall u \in \Lambda(S, \Gamma, \mathbb{R}^n).$$

Question (Plateau-Douglas problem)

- ▶ Is there an area minimizer $u_{\min} \in \Lambda(S, \Gamma, \mathbb{R}^n)$?
- ▶ If yes: Regularity of u_{\min} ?

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- ▶ Existence: **Yes!**

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 - ▶ However: In general no reasonable regularity!

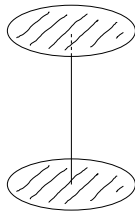
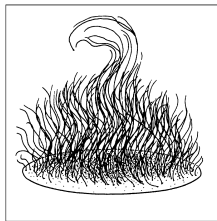


Figure taken from Dierkes-Hildebrandt-Sauvigny
"Minimal surfaces"

Energy minimization

- ▶ Set

$$\mathcal{R}(S) := \{\text{smooth Riemannian metrics on } S\}.$$

- ▶ For $g \in \mathcal{R}(S)$ and $u \in W^{1,2}(S, \mathbb{R}^n)$ define *Dirichlet energy*

$$E^2(u, g) := \frac{1}{2} \cdot \int_S |du|_g^2 \, dA_g.$$

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 - $E^2(u, g) \leq E^2(u, h)$ for all $h \in \mathcal{R}(S)$.
 - $u: (S, g) \rightarrow \mathbb{R}^n$ is *weakly conformal*.

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- ▶ So (u_{\min}, g_{\min}) energy minimizing pair
- ↪ $u_{\min}: (S, g_{\min}) \rightarrow \mathbb{R}^n$ is harmonic and weakly conformal

Existence of energy minimizers

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In general: No!

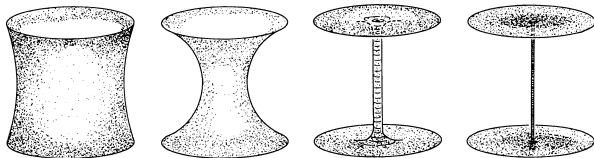
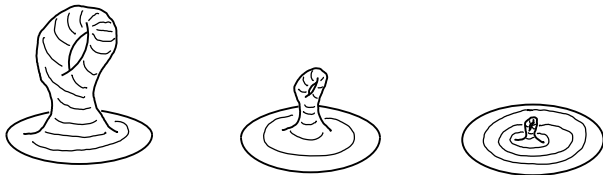
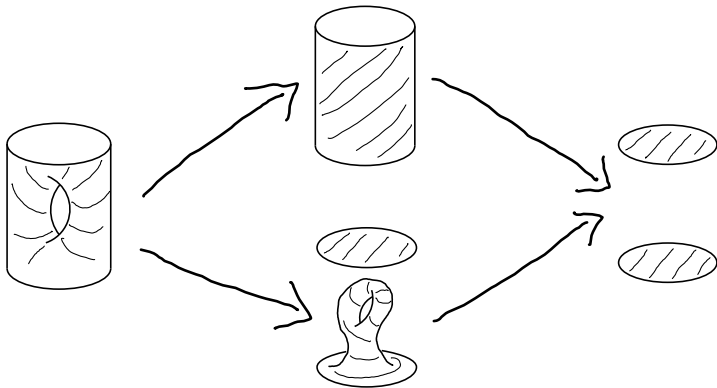


Figure taken from Dierkes-Hildebrandt-Tromba "Global analysis of minimal surfaces"



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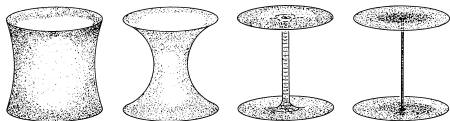
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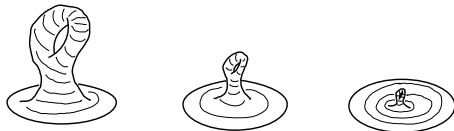
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\rightsquigarrow Douglas condition violated for $S^* = \mathbb{D} \sqcup \mathbb{D}$.



\rightsquigarrow Douglas condition violated for $S^* = \mathbb{D}$.

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Theorem (Douglas '39)

If Douglas condition holds $\rightsquigarrow \exists (u_{\min}, g_{\min})$ energy minimizing.

Corollary

If Douglas condition holds

$\rightsquigarrow \exists$ area minimizer $u_{\min} \in \Lambda(S, \Gamma, \mathbb{R}^n)$, and

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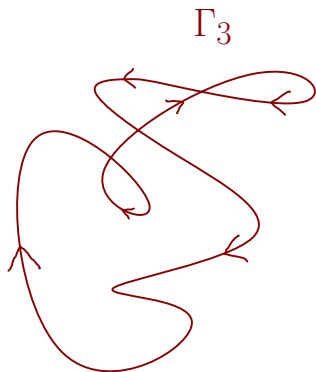
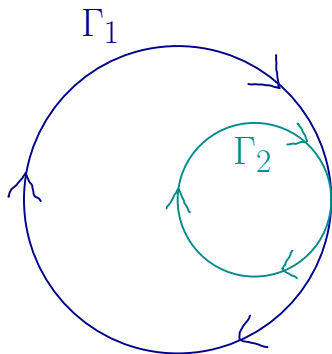
- ▶ S compact, connected, orientable surface with k boundary components.
- ▶ $\Gamma \subset \mathbb{R}^n$ a configuration of k disjoint Jordan curves of finite length.

Plateau-Douglas problem for singular boundary values

Now:

- ▶ S compact, connected, orientable surface with k boundary components.
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$$\Gamma = (\Gamma_1, \Gamma_2, \Gamma_3)$$



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If Douglas-condition holds

$\rightsquigarrow \exists$ 'reasonable' area minimizer $u_{\min} \in \Lambda(S, \Gamma, \mathbb{R}^n)$?

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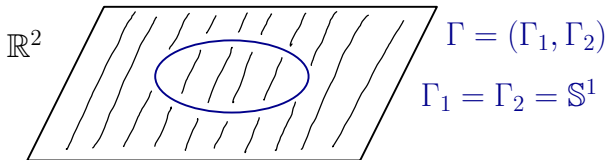
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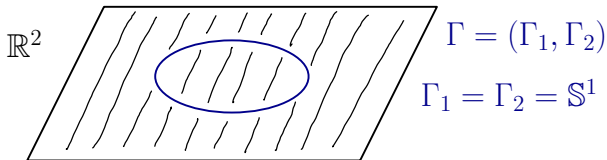
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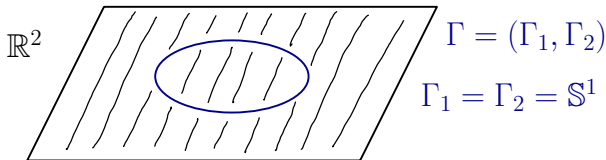
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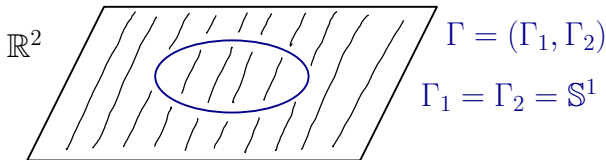
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- ▶ If (u_{\min}, g_{\min}) energy minimizing for (S, Γ)
 - $\rightsquigarrow u_{\min}$ weakly conformal + $\text{Area}(u_{\min}) = 0$
 - $\rightsquigarrow u_{\min}$ constant \nexists

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$\rightsquigarrow \exists$ 'reasonable' area minimizer $u_{\min} \in \Lambda(S, \Gamma, \mathbb{R}^n)$?

Hass ('91):

- "Yes" for $S = \mathbb{D}$
- u_{\min} is continuous
- Caution: In general $u_{\min} \notin \Lambda(\mathbb{D}, \Gamma, \mathbb{R}^n)$

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- If additionally "condition of adhesion" holds
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Theorem (C.-Fitzi, 2020)

Yes!

- ▶ u_{\min} is Hölder continuous.
- ▶ There is $g \in \mathcal{R}(S)$ such that $u_{\min}: (S, g) \rightarrow \mathbb{R}^n$ is harmonic and weakly conformal on $S \setminus u^{-1}(\Gamma)$.
- ▶ If Γ is $C^2 \rightsquigarrow u_{\min}$ locally Lipschitz on $S \setminus \partial S$.

3. Plateau-Douglas problem in singular spaces

Geometric analysis in metric spaces

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- ▶ $u_{\min} \in \Lambda(S, \Gamma, X)$ is called *area minimizer* if

$$\text{Area}(u_{\min}) \leq \text{Area}(u) \quad ; \forall u \in \Lambda(S, \Gamma, X).$$

Question

If Douglas-condition holds
 $\rightsquigarrow \exists$ 'reasonably regular' area minimizer $u_{\min} \in \Lambda(S, \Gamma, X)$?

Energy in the metric setting

- ▶ $(u_{\min}, g_{\min}) \in \Lambda(S, \Gamma, X) \times \mathcal{R}(S)$ is called *energy minimizing* pair if

$$E_+^2(u_{\min}, g_{\min}) \leq E_+^2(u, g) \quad ; \forall u \in \Lambda(S, \Gamma, X); \forall g \in \mathcal{R}(S).$$

- ▶ For $u \in W^{1,2}(S, X)$ in general

$$\text{Area}(u) \neq \inf_{g \in \mathcal{R}(S)} E_+^2(u, g)$$

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- ▶ Let $u \in W^{1,2}(S, \mathbb{R}^n)$ and $g \in \mathcal{R}(S)$. TFAE:

- $E_+^2(u, g) \leq E_+^2(u, h)$ for all $h \in \mathcal{R}(S)$.
- $u: (S, g) \rightarrow \mathbb{R}^n$ is weakly conformal.

- ▶ We say that $u: (S, g) \rightarrow X$ is *infinitesimally isotropic* if

$$E_+^2(u, g) \leq E_+^2(u, h) \quad ; \forall h \in \mathcal{R}(S).$$

Plateau-Douglas problem in singular spaces

- ▶ S compact, connected, orientable surface with k boundary components.
- ▶ $X = (X, d)$ a proper metric space.
- ▶ $\Gamma \subset X$ a configuration of k disjoint Jordan curves of finite length.

Theorem (C.-Fitzi, 2020)

If Douglas-condition holds, then

↪ \exists area minimizer $u_{\min} \in \Lambda(S, \Gamma, X)$, and

- $\exists g \in \mathcal{R}(S)$ s.t. $u_{\min}: (S, g) \rightarrow X$ is infinitesimally isotropic.

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- ▶ Previous results:
 - The case $S = \mathbb{D}$ (Lytchak-Wenger 2016)
 - For spaces X which satisfy a local quadratic isoperimetric inequality (Fitzi-Wenger 2020)
- ▶ New e.g. for general complete Riemannian manifolds X .

4. Proof sketch

Idea

- ▶ $\Gamma \subset \mathbb{R}^n$ a configuration of closed curves.

Theorem 1

If Douglas-condition holds $\rightsquigarrow \exists$ 'reasonable' area minimizer $u_{\min} \in \Lambda(S, \Gamma, \mathbb{R}^n)$

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Idea for Thm. 2 \rightsquigarrow Thm. 1:

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- ▶ Construct \tilde{Y} singular space and $\tilde{\Gamma} \subset \tilde{Y}$ regular configuration s.t.

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- ▶ Apply Theorem 2 to solve PD problem for $(\tilde{Y}, \tilde{\Gamma})$. □

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- ▶ W.l.o.g. $\Gamma = (\Gamma_1, \dots, \Gamma_k)$ with $\ell(\Gamma_i) \leq 2\pi$.

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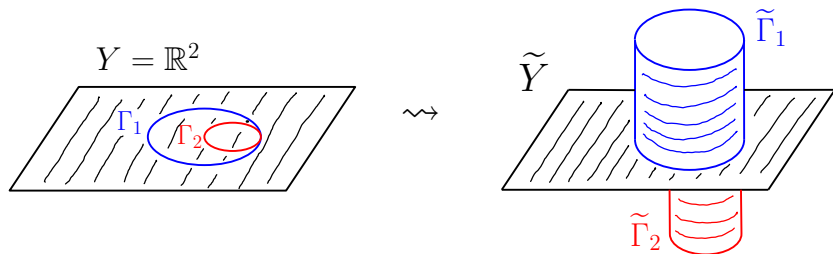
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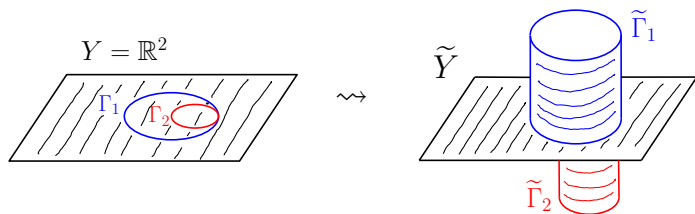
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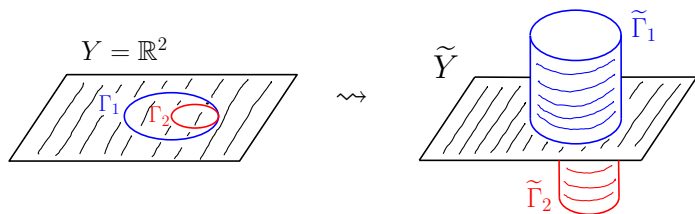


The proof



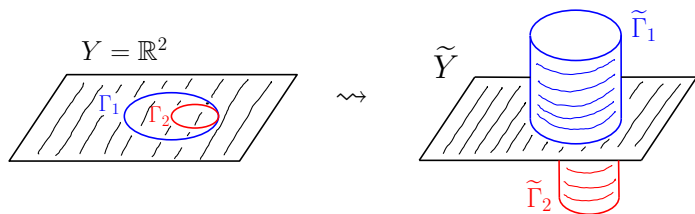
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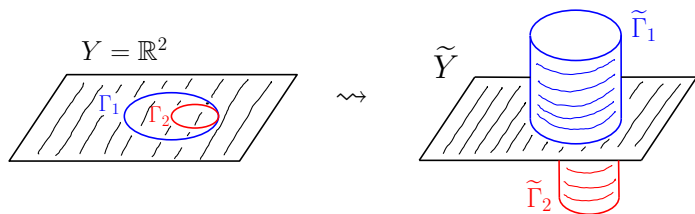
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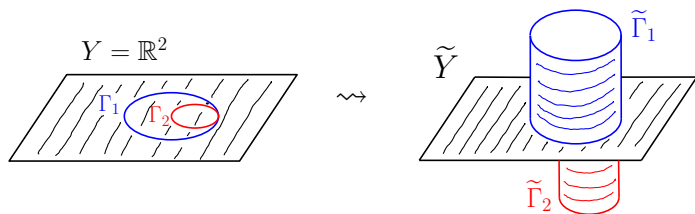
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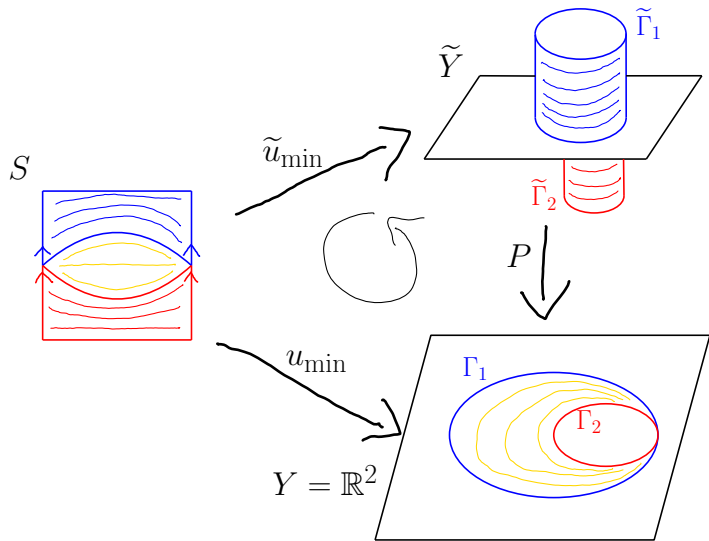
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Sketch of the argument



Proof of Theorem 2

- ▶ X a proper metric space,
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Theorem 2

If Douglas-condition holds $\rightsquigarrow \exists$ 'reasonable' area minimizer $u_{\min} \in \Lambda(S, \Gamma, X)$

Theorem (Fitzi-Wenger, 2020)

Theorem 2 holds when X satisfies a local quadratic isoperimetric inequality

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- $\rightsquigarrow u := \lim_{n \rightarrow \infty} u_n$ is solution to PD problem (Γ, X) □

Thank You!