Maximal metric surfaces

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- Analytic answer: A Riemannian manifold is a pair (M, g) where M is a differentiable manifold and $g \colon \mathrm{T}M \oplus \mathrm{T}M \to \mathbb{R}$ is a smooth function s.t. $g_p := g|_{\mathrm{T}_pM \oplus \mathrm{T}_pM}$ is a scalar product for every $p \in M$.

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- A Riemannian metric g on M determines a metric $d_g \colon M \times M \to \mathbb{R}$ by

$$\ell_g(\gamma) := \int_0^1 \sqrt{g_{\gamma(t)}\left(\gamma'(t), \gamma'(t)\right)}\, \mathrm{d}t$$

  for every absolutely continuous curve $\gamma \colon [0, 1] \to M$ and

$$d_g(x, y) := \inf_{\gamma \colon x \rightsquigarrow y} \ell_g(\gamma).$$

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Questions

- Let X be a metric space homeomorphic to M.
- Can we associate to X some sort of analytic data on M?
- If yes, how close is the connection between analysis and geometry?

Analytic data of bi-Lipschitz spheres

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- By Kirchheim ('94) for a.e. $p \in \mathbb{S}^2$ there is a seminorm $\mathrm{md}\varphi_p$ on $\mathrm{T}_p\mathbb{S}^2$ s.t.

$$\lim_{\substack{v \to 0 \\ v \in \mathrm{T}_p\mathbb{S}^2}} \frac{d\left(\varphi(\exp_p(v)), \varphi(p)\right) - \mathrm{md}\varphi_p(v)}{|v|} = 0.$$

Weak Finsler structures and the De Cecco–Palmieri metric

- A *weak Finsler structure* on $\mathbb{S}^2$ is a measurable function $F\colon \mathrm{T}\mathbb{S}^2 \to [0, \infty)$ s.t. $F_p := F|_{\mathrm{T}_p\mathbb{S}^2}$ is a seminorm for a.e. $p \in \mathbb{S}^2$.

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- The *F-length* of an absolutely continuous curve $\gamma\colon [0,1] \to \mathbb{S}^2$ is given by

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- One is tempted to define a metric d_F on $\mathbb{S}^2$ by

$$d_F(x,y) := \inf_{\gamma\colon x \rightsquigarrow y} \ell_F(\gamma).$$

- However, this e.g. needs not be symmetric.

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- An absolutely continuous curve $\gamma \subset \mathbb{S}^2$ is called *transversal* to $E \subset \mathbb{S}^2$ if

$$\mathcal{H}^1\left(\mathrm{im}(\gamma) \cap E\right) = 0.$$

- In this case we write $\gamma \perp E$.

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- One can define a (pseudo)metric d_F on $\mathbb{S}^2$ by

$$d_F(x,y) := \sup_{\substack{E \subset \mathbb{S}^2 \\ \mathcal{H}^2(E)=0}} \ \inf_{\substack{\gamma\colon x \rightsquigarrow y \\ \gamma \perp E}} \ell_F(\gamma).$$

$$\operatorname{ess\,inf}(f) := \sup_{|E|=0} \ \inf_{x \in \mathbb{R}\setminus E} f(x)$$

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- If $F_p = \overline{F}_p$ for a.e. $p \in \mathbb{S}^2$ then $d_F = d_{\overline{F}}$.

Coordinate independence under bi-Lipschitz changes

- Let F be a weak Finsler structure on $\mathbb{S}^2$.
- Let $\Phi\colon \mathbb{S}^2 \to \mathbb{S}^2$ be a bi-Lipschitz homeomorphism.

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- By Rademacher's theorem we can define $\Phi^* F$ on $\mathbb{S}^2$ by setting

  $$(\Phi^* F)_p := F_{\Phi(p)} \circ \mathrm{d}\Phi_p$$

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- $\mathcal{H}^2(E) = 0$ if and only if $\mathcal{H}^2(\Phi(E)) = 0$.
- $\gamma \subset \mathbb{S}^2$ is absolutely continuous if and only if $\Phi \circ \gamma$ is absolutely continuous.
- $\gamma \perp E$ if and only if $\Phi \circ \gamma \perp \Phi(E)$.
- There $E \subset \mathbb{S}^2$ s.t. $\mathcal{H}^2(E) = 0$ and

$$\begin{aligned} \ell_{\Phi^* F}(\gamma) &= \int_0^1 \left(F_{(\Phi\circ\gamma)(t)} \circ \mathrm{d}\Phi_{\gamma(t)}\right)(\gamma'(t))\,\mathrm{d}t \\ &= \int_0^1 F_{(\Phi\circ\gamma)(t)}\left((\Phi\circ\gamma)'(t)\right)\,\mathrm{d}t = \ell_F(\Phi\circ\gamma) \end{aligned}$$

  for every γ with $\gamma \perp E$.
- Thus $\Phi\colon \left(\mathbb{S}^2, d_{\Phi^* F}\right) \to \left(\mathbb{S}^2, d_F\right)$ is an isometry.

Analytic data of bi-Lipschitz spheres

- Let $\varphi \colon \mathbb{S}^2 \to X$ be a bi-Lipschitz homeomorphism.
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- We call the weak Finsler structure $F^X := \mathrm{md}\varphi$ the *analytic data of X*.
- This is well defined in the sense that if $\psi\colon \mathbb{S}^2 \to X$ is another bi-Lipschitz homeomorphism, then $\psi = \varphi \circ \Phi$ for a bi-Lipschitz homeomorphism $\Phi\colon \mathbb{S}^2 \to \mathbb{S}^2$ and

$$\mathrm{md}\psi = \mathrm{md}(\varphi\circ\Phi) = \mathrm{md}\varphi \circ \mathrm{d}\Phi = \Phi^*\mathrm{md}\varphi.$$

- In particular, $X^{DCP} := \left(\mathbb{S}^2, d_{F^X}\right)$ depends, up to isometry, only on X.

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- There is $E \subset \mathbb{S}^2$ with $\mathcal{H}^2(E) = 0$ s.t.

$$\ell_{F^X}(\gamma) = \int_0^1 \mathrm{md}\varphi_{\gamma(t)}(\gamma'(t))\,\mathrm{d}t = \int_0^1 \mathrm{md}(\varphi \circ \gamma)_t(1)\,\mathrm{d}t = \ell(\varphi \circ \gamma)$$

$\geq d(\varphi(x), \varphi(y$

  for every $\gamma \perp E$.
- Hence $\varphi\colon X^{DCP} \to X$ is 1-Lipschitz.

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Theorem (De Cecco–Palmieri '95)

- X^{DCP} is bi-Lipschitz equivalent to $\mathbb{S}^2$.
- X^{DCP} is analytically equivalent to X.
- If Y is analytically equivalent to X then exists a 1-Lipschitz homeomorphism $f\colon X^{DCP} \to Y$.
- These properties uniquely characterize X^{DCP} up to isometry.

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$$\ell_{\Phi^* F}(\gamma) = \int_0^1 \left(F_{(\Phi\circ\gamma)(t)} \circ \mathrm{apd}\Phi_{\gamma(t)}\right)(\gamma'(t))\ \mathrm{d}t$$
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  for every γ with $\gamma \perp E$.

  $\Phi\circ\gamma$ abs. cont.?
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- If $\mathcal{H}^2(X) < \infty$ then $\varphi \in N^{1,2}(\mathbb{S}^2, X)$. (Tyson '00)
- For a.e. $p \in \mathbb{S}^2$ there is a seminorm $\mathrm{apmd}\varphi_p$ on $\mathrm{T}_p\mathbb{S}^2$ s.t.

$$\operatorname*{ap\,lim}_{\substack{v \to 0 \\ v \in \mathrm{T}_p\mathbb{S}^2}} \frac{d\left(\varphi(\exp_p(v)), \varphi(p)\right) - \mathrm{apmd}\varphi_p(v)}{|v|} = 0.$$

- If $\psi\colon \mathbb{S}^2 \to X$ is another quasisymmetric homeomorphism, then $\Phi = \varphi^{-1} \circ \psi$ is also only quasisymmetric.
- Since Φ is only quasisymmetric we do not get a well-defined space X^{DCP}

Modulus and Newtonian-Sobolev spaces

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- For $p < \infty$ the *p-modulus* of Γ is given by

$$\mathrm{Mod}_p(\Gamma) := \inf \left\{ ||\rho||_{L^p}^p \mid \rho \colon \mathbb{S}^2 \to [0, \infty] \ \wedge \ \int_\gamma \rho \geq 1 \ ; \ \forall \gamma \in \Gamma \right\}.$$

- The ∞-*modulus* of Γ is given by

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- A function $f \in L^p(\mathbb{S}^2, X)$ lies in $N^{1,p}(\mathbb{S}^2, X)$ if there is $\rho \in L^p(\mathbb{S}^2)$ such that $f \circ \gamma$ is absolutely continuous and satisfies

$$\ell(f \circ \gamma) \leq \int_\gamma \rho$$

  for p-almost every curve γ in $\mathbb{S}^2$.
- The minimal such ρ will be called the *minimal p-weak upper gradient* and denoted ρ_f.

Canonical quasisymmetric parametrizations

$\mathcal{H}^d(B(x,r)) \simeq r^d$

- Let X be Ahlfors 2-regular, linearly locally connected and homeomorphic to $\mathbb{S}^2$.

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- Denote by $\Lambda(X)$ the set of homeomorphisms in $N^{1,2}(\mathbb{S}^2, X)$.
- For $\varphi \in \Lambda(X)$ the *Reshetnyak energy* is defined by

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Theorem (Bonk–Kleiner '02, Lytchak–Wenger '20)

- There is $\varphi \in \Lambda(X)$ of least energy $E_+^2(\varphi)$.
- Every such φ is a quasisymmetric homeomorphism $\mathbb{S}^2 \to X$.
- If $\psi \in \Lambda(X)$ is another energy minimizing homeomorphism, then $\psi = \varphi \circ \Phi$ where $\Phi\colon \mathbb{S}^2 \to \mathbb{S}^2$ is a conformal diffeomorphism.

Analytic data of quasispheres

- Let X be Ahlfors 2-regular and linearly locally connected.
- Let $\varphi\colon \mathbb{S}^2 \to X$ be an energy-minimizing quasisymmetric homeomorphism.
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- We cannot deduce that $\varphi\colon X^{DCP} \to X$ is 1-Lipschitz.
- We do not know about a natural characterization of X^{DCP}.

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- Let Γ be a family of absolutely continuous curves in $\mathbb{S}^2$
- Then $\mathrm{Mod}_\infty(\Gamma) = 0$ if and only if exists $E \subset \mathbb{S}^2$ s.t. $\mathcal{H}^2(E) = 0$ and $\gamma \not\subset E$ for all $\gamma \in \Gamma$.

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- Problem: $\mathrm{Mod}_2(\{\gamma\colon x \rightsquigarrow y\}) = 0$ and hence $\widehat{d}_F(x,y) = \infty$ when $x \neq y$.

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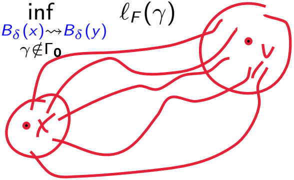

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- Since $\ell_{FX}(\gamma) = \ell(\varphi \circ \gamma)$ for 2-almost every curve it follows that

$$\widehat{d}_F(x,y) \geq d(\varphi(x), \varphi(y)).$$

- It is however e.g. not obvious that $\widehat{d}_{FX}$ satisfies the triangle inequality.

Characterization of $\widehat{X}$

- Let X and Y be Ahlfors 2-regular quasispheres.
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- *Volume rigidity.* If Y is an Ahlfors 2-regular quasisphere and $f\colon Y \to \widehat{X}$ is an area preserving 1-Lipschitz homeomorphism, then f is an isometry.

Thank You!